Circuit Analysis I
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CHAPTER 7 *RC* and *RL* Circuits – Time Domain Analysis

7.1 Introduction

So far, we have considered resistors, capacitors and inductors individually. We are now prepared to consider circuits that contain various combinations of two or three of these elements.

We noted an important property of capacitors and inductors: ability to store energy. We are now ready to determine the currents and voltages that arise in the circuit when energy is either absorbed or released by an energy-storing element.

We shall examine two types of simple circuits: a circuit comprising of resistors and capacitor (*RC circuit*) and a circuit comprising of resistors and an inductor (*RL circuit*). The analysis of *RC* and *RL* circuits will lead to first-order differential equations. Hence, the circuits are collectively known as *first-order circuits*.

In addition to there being two types of first-order circuits (*RC* and *RL*), there are two ways to excite the circuits.

The first way is by initial conditions of the storage elements in the circuits. In these so-called *source-free circuits*, we assume that the energy is initially stored in the capacitor or in the inductor in the form of the initial voltage or current. When the capacitor or inductor is suddenly disconnected from the dc source (that established the initial conditions) and connected to a resistive network, it will give rise to voltages and currents in that network.

The voltages and currents that arise in that configuration are referred to as the ***natural response*** of the circuit.

The second way of exciting first-order circuits is by independent sources. For now, we will consider dc sources only. That is, the voltages and currents that arise in the circuit are due to the sudden application of a dc voltage or current. This response is referred to as the ***step response***.

7.2 Formulating *RC* and *RL* Circuit Equations

In general, *RC* and *RL* circuits contain sources, resistors and a single capacitor or a single inductor (which may represent the equivalent capacitance or inductance of a combination of capacitors or inductors).

Any such circuit can be divided into two parts: (1) resistors and sources, and (2) the dynamic element (capacitor or inductor), as shown in Fig. 7.1-1.

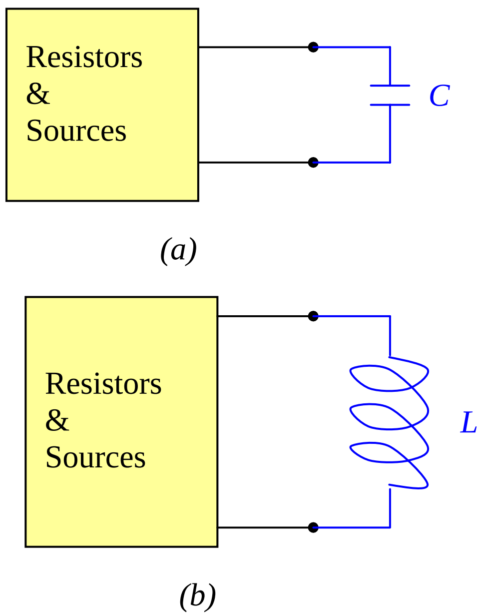


Figure 7.1-1 First-order circuits: (a) *RC* circuit, (b) *RL* circuit

To formulate the equation governing either of these circuits, we replace the resistors and sources by their Thevenin and Norton equivalents, as shown in Fig. 7.1-2.

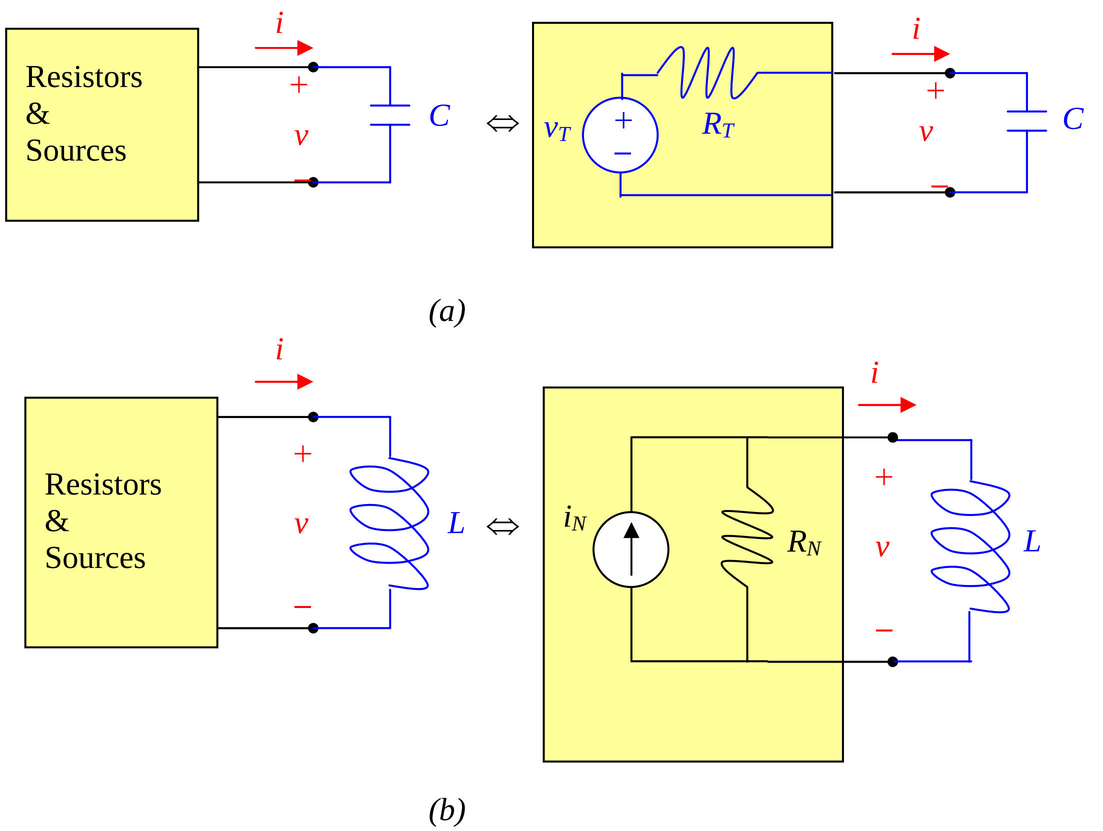


Figure 7.1-2 First-order circuits and their equivalents: (a) *RC* circuit, (b) *RL* circuit

RC circuit on the RHS of Fig. 7.1-2(a) is governed by the following constraint

$$R_T i(t) + v(t) = v_T(t) \tag{7.2-1}$$

The capacitor *i*-*v* constraint is

$$i(t) = C\frac{dv(t)}{dt} \tag{7.2-2}$$

Substituting Eq. (7.1-2) into Eq. (7.1-1) yields

$$\boxed{R_T C\frac{dv(t)}{dt} + v(t) = v_T(t)} \tag{7.2-3}$$

This equation governs the *RC* circuit. The unknown is the capacitor voltage *v(t)* that is referred to as the *state variable*. Mathematically, Eq. (7.1-3) is a first-order linear differential equation with constant coefficients.

♦

RL circuit on the RHS of Fig. 7.1-2(b) is governed by the following constraint

$$\frac{v(t)}{R_N} + i(t) = i_N(t) \tag{7.2-4}$$

The inductor *i*-*v* constraint is

$$v(t) = L\frac{di(t)}{dt} \tag{7.2-5}$$

Substituting Eq. (7.1-5) into Eq. (7.1-4) yields

$$\boxed{\frac{L}{R_N}\frac{di(t)}{dt} + i(t) = i_N(t)} \tag{7.2-6}$$

This equation governs the *RL* circuit. The unknown is the inductor current *i(t)* that is referred to as the *state variable*.

Mathematically, Eq. (7.1-6) is a first-order linear differential equation with constant coefficients.

Note that the mathematical from of the *RC* and *RL* circuit equations is the same. That is, both equations can be written in the form:

$$\boxed{A\frac{dx(t)}{dt} + x(t) = f(t)} \tag{7.2-7}$$

where *x(t)* is the state-variable, *f(t)* is the forcing function, and *A* is a real constant.

This means that the mathematical form of the solutions will be the same.

7.3 Source-Free RC Circuit – Natural Response

The source-free *RC* circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor causes the current to flow in the circuit and is gradually dissipated in the resistors.

The natural response of an *RC* circuit is developed from the circuit shown in Fig. 7.3-1

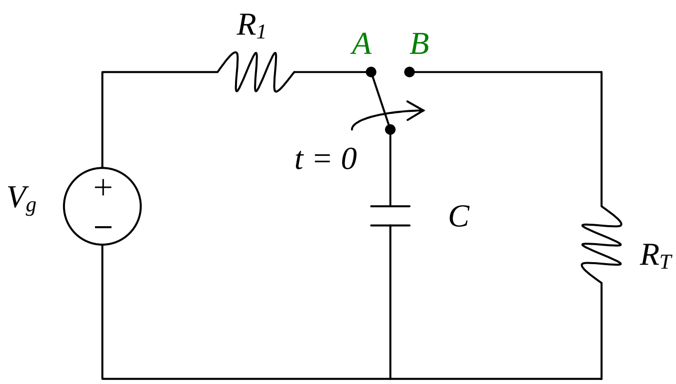


Figure 7.3-1 An *RC* circuit

Note that R_T represents the Thevenin resistance of the circuit to which the capacitor is connected.

When the switch is in position *A* (has been for a long time), under the dc condition the capacitor acts as an open circuit and the voltage across it is V_g.

At $t = 0$ the switch moves from position *A* to *B*; the resulting *RC* circuit is shown in Fig. 7.3-2.

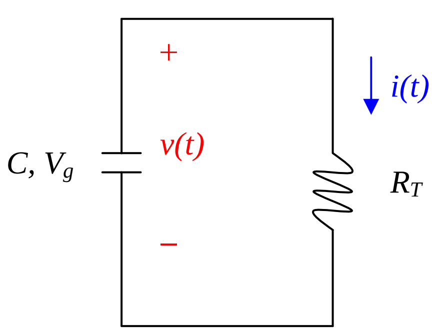


Figure 7.3-2 A source-free *RC* circuit

Thus, the circuit consists of a series combination of a resistor and initially charged capacitor. Our objective is to determine the voltage *v(t)* across the capacitor.

We select the capacitor voltage as the response in order to take advantage of the idea that the capacitor voltage cannot change instantaneously.

Since the voltage across the capacitor may not change instantaneously, it follows that

$$V_g = V(0^-) = V(0) = V(0^+) = V_0 \tag{7.3-1}$$

That is, the initial voltage on the capacitor in Fig. 7.3-2 is V_g.

$$v(0) = V_g = V_0 \tag{7.3-2}$$

The corresponding value of the energy stored in the capacitor is

$$w(0) = \frac{1}{2} C V_0^2 \tag{7.3-3}$$

Applying KVL to circuit in Fig. 7.3-2 produces

$$-v(t) + R_T i = 0 \tag{7.3-4}$$

The capacitor *i-v* constraint is

$$i(t) = -C \frac{dv(t)}{dt} \tag{7.3-5}$$

Substituting Eq. (7.3-5) into Eq. (7.3-4) yields

$$\boxed{R_T C \frac{dv(t)}{dt} + v(t) = 0} \tag{7.3-6}$$

This first-order, linear, constant-coefficient, homogeneous differential equation can be solved using any of the several techniques. We will solve it using the method of the separation of variables.

To solve this equation, rearrange the terms as

$$\frac{dv(t)}{dt} = -\frac{v(t)}{R_T C} \qquad (7.3\text{-}7)$$

or

$$\frac{dv}{v} = -\frac{1}{R_T C} dt \qquad (7.3\text{-}8)$$

Integrating both sides, we get

$$\ln v = -\frac{t}{R_T C} + A \qquad (7.3\text{-}9)$$

where *A* is the integration constant.

Solving for *v(t)* produces

$$v(t) = e^{\left(-t/R_T C\right)+A} = e^{-t/R_T C} e^{A} \qquad (7.3\text{-}10)$$

or

$$v(t) = Be^{-t/R_T C} \qquad (7.3\text{-}11)$$

From the initial condition

$$v(0) = V_0 \quad \Rightarrow \quad V_0 = Be^0 = B \qquad (7.3\text{-}12)$$

Hence,

$$\boxed{v(t) = V_0 e^{-t/R_T C}, \quad t \geq 0} \qquad (7.3\text{-}13)$$

This shows that the voltage response of the *RC* circuit is an exponential decay of the initial voltage.

Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the ***natural response*** of the circuit.

The natural response is also called the ***zero-input response***. The natural response is illustrated graphically in Fig. 7.3-3.

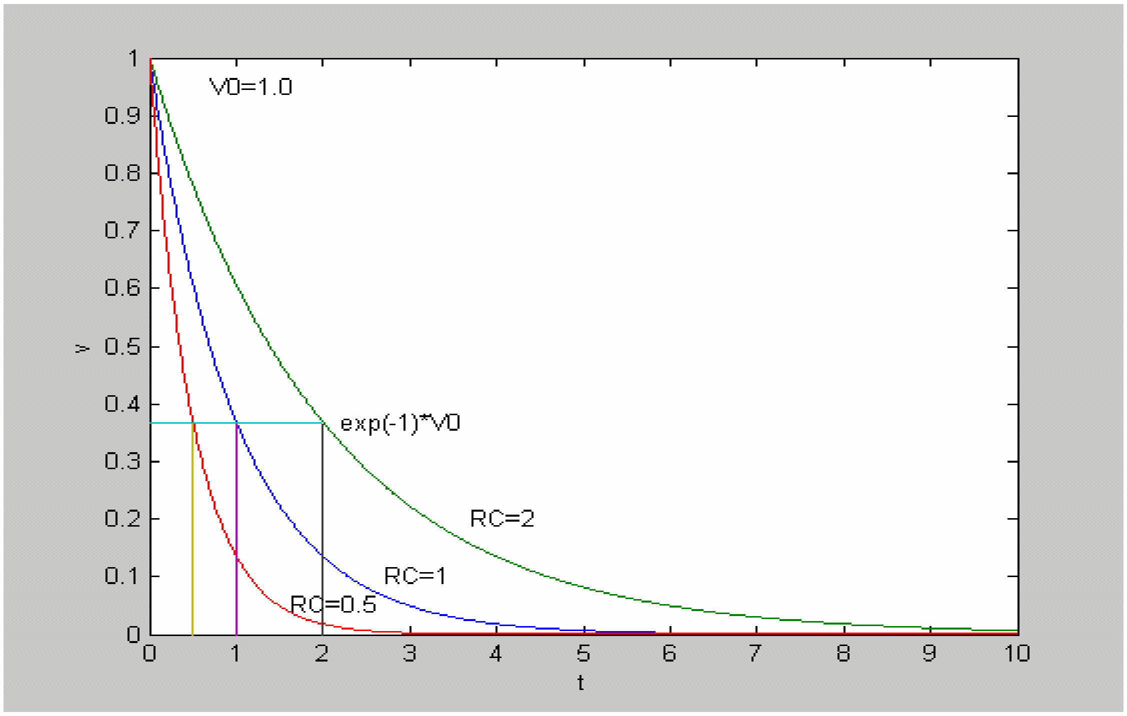


Figure 7.3-3 Zero-input response of an *RC* circuit

Note that at $t = 0$ we have the correct initial condition. The exponential response starts at $v(0) = V_0$ and then decays to zero as $t \to \infty$.

The rapidity with which the voltage decreases is expressed in terms of the ***time constant***, denoted by the lower case Greek letter τ.

*The **time constant** of a circuit is the time required for the response to decay by a factor e^{-1} of its initial value.*

This definition implies that at $t = \tau$, Eq. (7.3-13) becomes

$$e^{-1}V_0 = V_0 e^{-\tau / R_T C} \tag{7.3-14}$$

or

$$\tau = R_T C \qquad \left[\Omega F = \frac{V}{A}\frac{As}{V} = s \right] \tag{7.3-15}$$

Note that the time constant has the units of seconds. In terms of the time constant, Eq. (7.3-13) can be written as

$$v(t) = V_0 e^{-t/\tau} \tag{7.3-16}$$

After five time constants, the voltage across the capacitor is less than 1% of its original value. Thus, it is customary to assume that the capacitor is fully discharged (or charged) after five time constants.

Observe that the smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response.

With the voltage *v(t)* given by Eq. (7.3-16) we can determine the *current* through the resistor as

$$i(t) = \frac{v(t)}{R_T} = \frac{V_0}{R_T} e^{-t/\tau} \tag{7.3-17}$$

The *power* dissipated in the resistor is

$$p(t) = v(t)i(t) = \frac{V_0^2}{R_T} e^{-2t/\tau} \tag{7.3-18}$$

The *energy* dissipated by the resistor from $t = 0$ up to time *t* is

$$
\begin{aligned}
w(t) &= \int_0^t p(t)dt = \int_0^t \frac{V_0^2}{R_T} e^{-2t/\tau} dt = \frac{V_0^2}{R_T}\left(-\frac{\tau}{2}\right) e^{-2t/\tau}\Big|_0^t \\
&= -\frac{V_0^2}{R_T}\frac{R_T C}{2}\left(e^{-2t/\tau} - 1\right) = \frac{CV_0^2}{2}\left(1 - e^{-2t/\tau}\right)
\end{aligned} \tag{7.3-19}
$$

Notice that as $t \to \infty$, $w(t) \to \frac{1}{2}CV_0^2$, which is the same as $w(0)$, the energy initially stored in the capacitor.

In summary, the key steps in working with a source-free *RC* circuit are:

(1) Determine *initial voltage* V_0, across the capacitor.

(2) Obtain *Thevenin equivalent resistance* R_T, at the terminals of the capacitor.

(3) Evaluate the *time constant* τ.

(4) Determine the *capacitor voltage v(t)*.

(5) Determine *other circuit variables* (capacitor current, resistor voltage).

B Step Response of an RC Circuit

Linear circuits are often characterized by applying step function and sinusoidal inputs. This section introduces step response of an *RC* circuit. Before proceeding with the analysis of a step response, let's review the step function.

The general step function is based on the ***unit step function*** defined as

$$
u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{3.12}
$$

Mathematically, the function *u(t)* has a jump discontinuity at $t = 0$. Fig. 3-3 shows the unit step function.

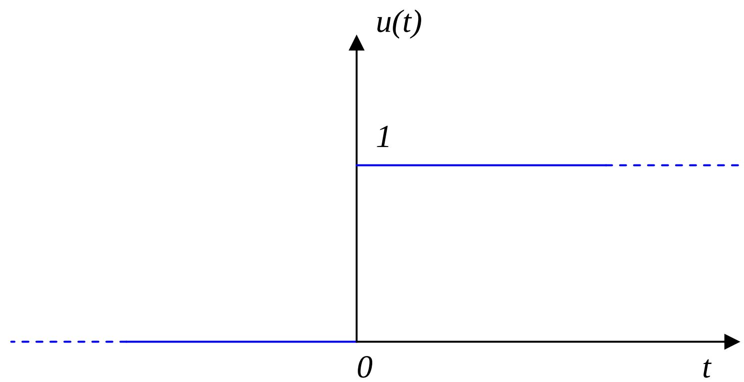


Figure 3-3 Unit step function

Strictly speaking, it is impossible to generate a true step function since signal variables like current and voltage cannot jump from one value to another in zero time. Practically speaking, we can generate very good approximations to the step functions. What is required is that the transition time be short compared with other response times in the circuit.

Multiplying *u(t)* by a constant *K* produces the ***step function***

$$Ku(t) = \begin{cases} 0 & t < 0 \\ K & t > 0 \end{cases} \tag{3.13}$$

shown in Figure 3-4.

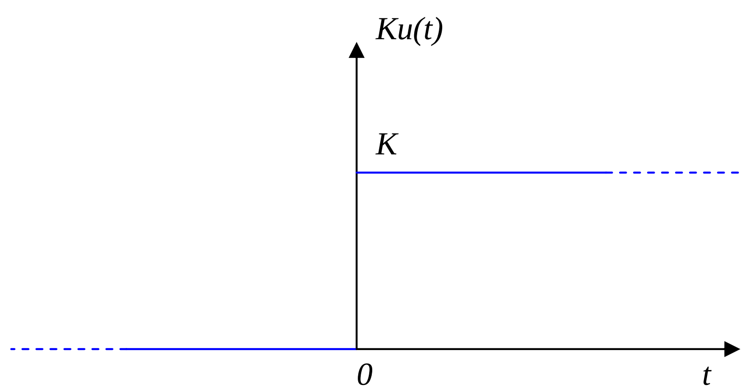


Figure 3-4 Step function

When the dc source of an *RC* circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is called a ***step response***.

The step response is the response of the circuit to a sudden application of a dc voltage or dc current source.

Consider the *RC* circuit shown in Fig. 3-5(a), which can be replaced by the circuit in Fig. 3-5(b), where V_T is a constant, dc source.

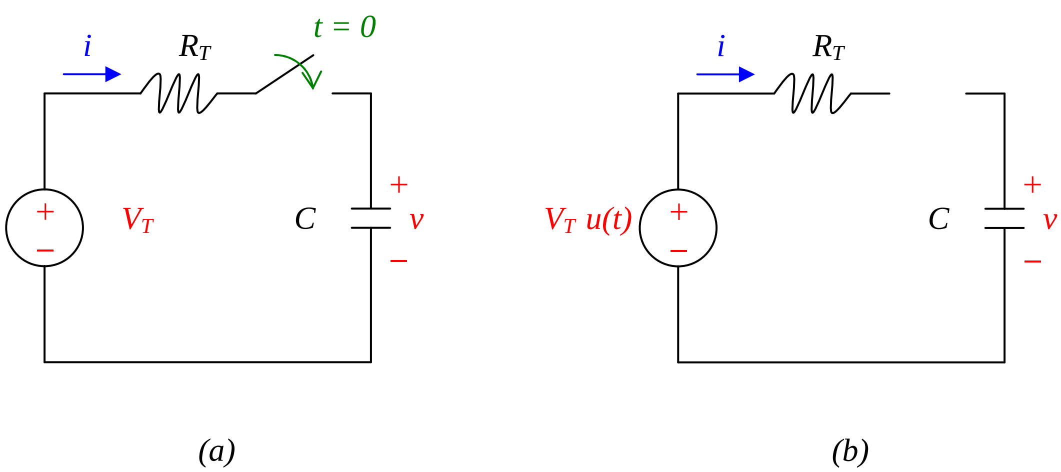


Figure 3-5 An *RC* circuit with voltage step input

Again, we select the capacitor voltage as the circuit response to be determined. We assume an initial voltage V_0 on the capacitor. Since the voltage across the capacitor cannot change instantaneously, we have

$$v(0^-) = v(0^+) = V_0 \tag{3.14}$$

where $v(0^-)$ is the voltage across the capacitor just before the switching and $v(0^+)$ is its voltage immediately after the switching.

Applying KVL to the circuit in Fig. 3-5(b) we obtain the differential equation

$$\boxed{R_T C \frac{dv}{dt} + v = V_T u(t)} \tag{3.15}$$

The total response is a function *v(t)* that satisfies this differential equation for $t > 0$ and meets the initial condition *v(0)*. Since $u(t) = 1$ for $t > 0$ we can write Eq. (3.15) as

$$R_T C \frac{dv(t)}{dt} + v(t) = V_T, \qquad t > 0 \tag{3.16}$$

Mathematics provides a number of approaches to solving this equation. We will solve it using the separation of variables. Rearranging terms gives

$$\frac{dv}{dt} = \frac{V_T - v}{R_T C} = -\frac{v - V_T}{R_T C} \tag{3.17}$$

or

$$\frac{dv}{v - V_T} = -\frac{dt}{R_T C} \tag{3.18}$$

Integrating both sides and introducing initial conditions

$$\ln(v - V_T)\Big|_{V_0}^{v(t)} = -\frac{t}{R_T C}\Bigg|_0^t \tag{3.19}$$

Thus,

$$\ln(v(t) - V_T) - \ln(V_0 - V_T) = -\frac{t}{R_T C} \tag{3.20}$$

or

$$\ln\frac{v - V_T}{V_0 - V_T} = -\frac{t}{R_T C} \tag{3.21}$$

Taking the exponential of both sides gives

$$\frac{v - V_T}{V_0 - V_T} = e^{-t/\tau}, \qquad \tau = R_T C \tag{3.22}$$

leading to

$$v - V_T = (V_0 - V_T)e^{-t/\tau} \tag{3.23}$$

or

$$v(t) = V_T + (V_0 - V_T)e^{-t/\tau}, \quad t \geq 0 \tag{3.25}$$

Assuming $V_T > V_0$, a plot of *v(t)* is shown in Fig. 3-6.

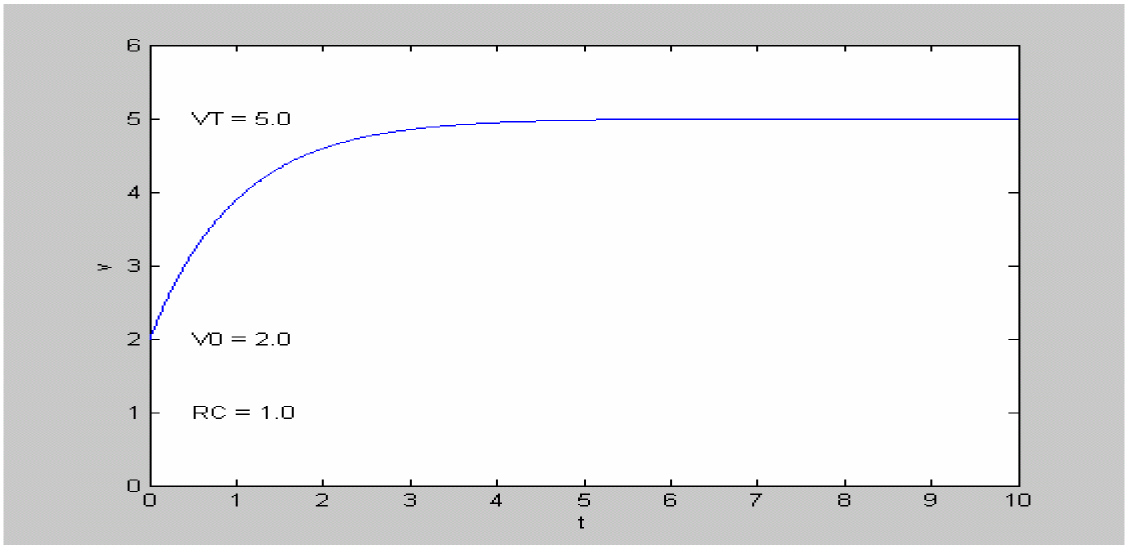


Figure 3-6 Step response of a first-order *RC* circuit with capacitor initially charged

This is known as a *complete response* of the *RC* circuit to a sudden application of a dc voltage source, assuming the capacitor was initially charged.

If we assume that the capacitor was initially uncharged, we set $V_0 = 0$ in (3.25) so that

$$v(t) = V_T\left(1 - e^{-t/\tau}\right), \quad t \geq 0 \tag{3.26}$$

Fig. 3-7 shows the plot of this waveform.

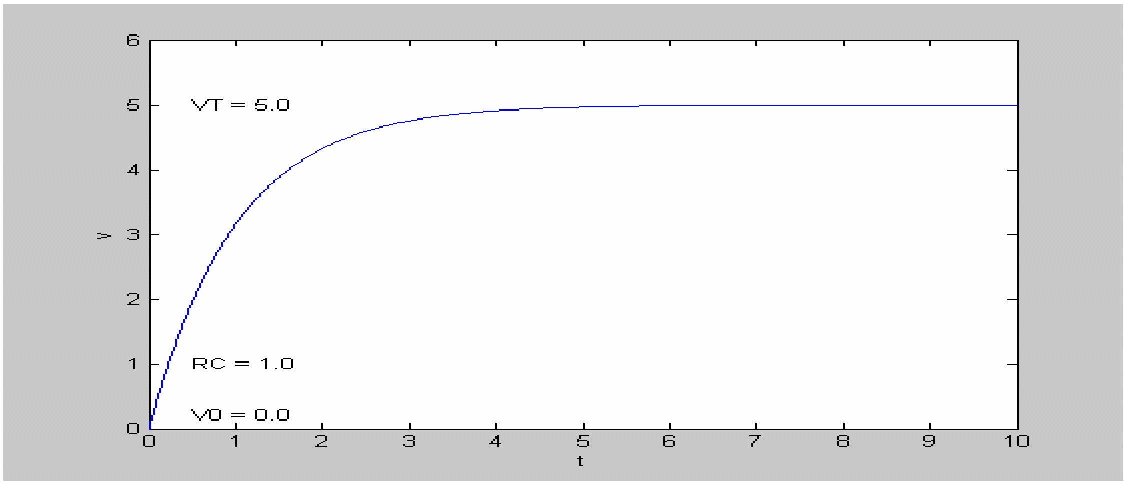


Figure 3-7 Step response of a first-order *RC* circuit with initially uncharged capacitor

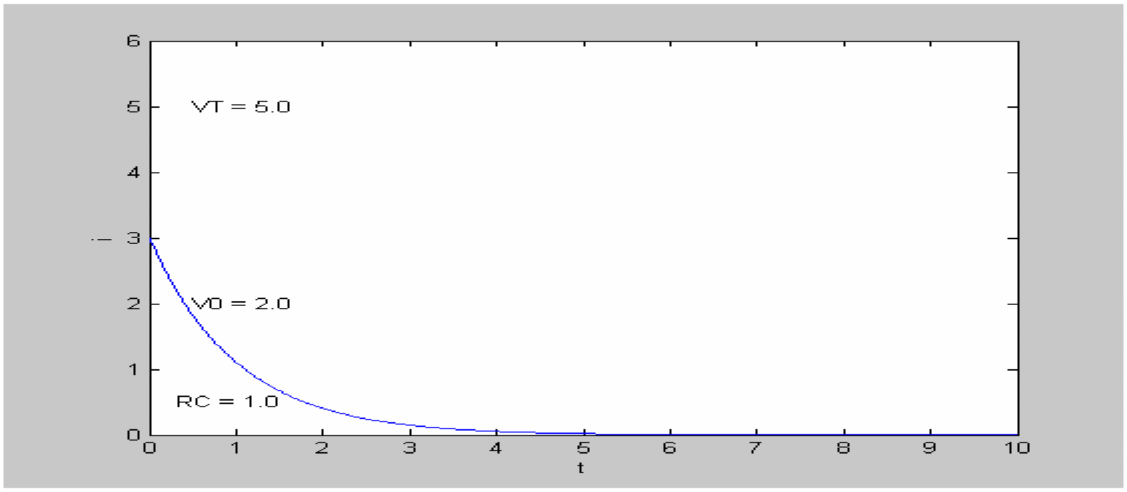
The current through the capacitor is obtained from (3.25) as

$$\begin{aligned} i(t) &= C\frac{dv}{dt} = C\frac{d}{dt}\left(V_T + (V_0 - V_T)e^{-t/\tau}\right) = C\frac{d}{dt}\left((V_0 - V_T)e^{-t/\tau}\right) \\ &= \frac{C(V_0 - V_T)}{-\tau}e^{-t/\tau} = \frac{V_T - V_0}{R_T}e^{-t/\tau} \end{aligned} \tag{3.27}$$

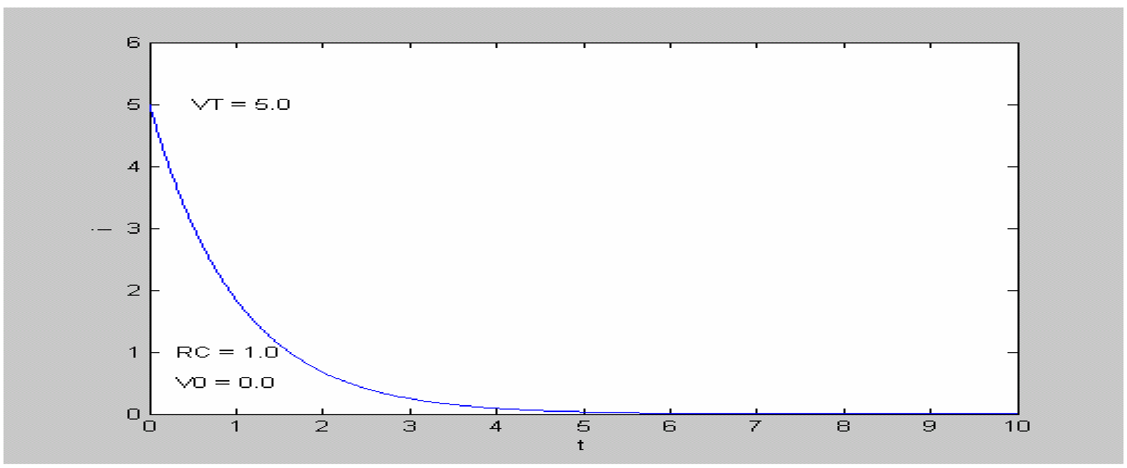
or

$$i(t) = \left(\frac{V_T}{R_T} - \frac{V_0}{R_T}\right)e^{-t/\tau}, \quad t \geq 0^+$$

Figs. 3-8(a) and (b) show the plots of *i(t)* for the initially charged and uncharged capacitor, respectively.



(a)



(b)

Figure 3-8 Current step response of an *RC* circuit: a) with initially charged capacitor, b) with initially uncharged capacitor

Let us reexamine Eq. (3.25), repeated here,

$$v(t) = V_T + (V_0 - V_T)e^{-t/\tau}, \quad t \geq 0 \tag{3.28}$$

It is evident that *v(t)* can be split into two components

$$v = v_{ss} + v_{tr} \tag{3.29}$$

where

$$v_{ss} = V_T \tag{3.30}$$

and

$$v_{tr} = (V_0 - V_T)e^{-t/\tau} \tag{3.31}$$

v_{tr} is the part of the response that will decay to zero with time, it is also called the *transient response* (because it is temporary).

v_{ss} is called the *steady-state response*, because it remains with time.

The complete response of the circuit is the sum of *the transient response and the steady-state response*.

We may thus rewrite (3.28) in the form:

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \tag{3.32}$$

♦

Now, let's return to Eq. (3.28) and rewrite it as:

$$v(t) = V_T + (V_0 - V_T)e^{-t/\tau} = V_0 e^{-t/\tau} + V_T\left(1 - e^{-t/\tau}\right)$$

It is evident that the total response may be separated into two parts: one due to the external input (forced response) and one due to the initial conditions (natural response).

Thus,

$$v = v_n + v_f$$

where

$$v_n(t) = V_0 e^{-t/\tau}$$

and

$$v_f(t) = V_T\left(1 - e^{-t/\tau}\right)$$

Total Response of an RL circuit

When the input to the *RL* circuit is a step function, we can write the Norton source as

$$i_N(t) = I_N u(t) \tag{4.1}$$

as shown in Fig. 4-1.

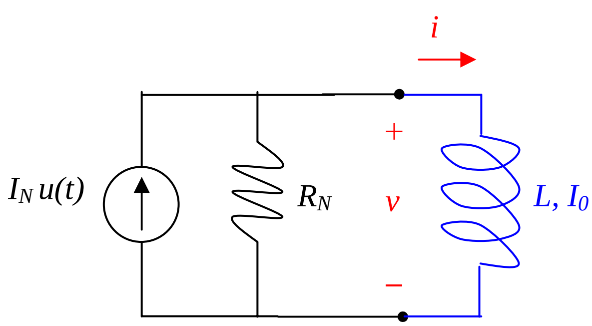


Figure 4-1 An *RL* circuit with a step input current

Our goal is to find the inductor current *i* as the circuit response. The circuit in Fig. 7.1-2(b) is governed by the following constraint

$$\frac{v(t)}{R_N} + i(t) = I_N, \quad t > 0 \tag{7.2-4}$$

The inductor *i*-*v* constraint is

$$v(t) = L\frac{di(t)}{dt} \tag{7.2-5}$$

Substituting Eq. (7.1-5) into Eq. (7.1-4) yields

$$\boxed{\frac{L}{R_N}\frac{di(t)}{dt} + i(t) = I_N} \tag{7.2-6}$$

The total response is a function *v(t)* that satisfies this differential equation for $t > 0$ and meets the initial condition *i(0).*

Now, recall the differential equation describing an RC circuit subjected to a step input.

$$R_T C \frac{dv(t)}{dt} + v(t) = V_T, \qquad t > 0$$

The mathematical form of the two equations is the same, thus the form of the solution has to be the same also.

The solution of an RC circuit was

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

Therefore, the solution of an RL circuit can be expressed as:

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

or, equivalently,

$$i(t) = I_N + (I_0 - I_N)e^{-t/\tau}$$

♦

Let the response be the sum of the natural current and the forced current,

$$i = i_n + i_f \qquad (4.2)$$

We know that the natural response is always a decaying exponential, that is

$$i_n = Ae^{-t/\tau}, \qquad \tau = \frac{L}{R} \qquad (4.3)$$

where *A* is a constant to be determined.

The forced response is the value of the current a long time after the switch in Fig. 4-1 is closed. We know that the natural response essentially dies out after five time constants. At that time, the inductor becomes a short circuit, and the forced response is

$$i_f = I_N \qquad (4.4)$$

The total response is, therefore,

$$i(t) = Ae^{-t/\tau} + I_N \tag{4.5}$$

We now determine the constant *A* from the initial value of *i*.

$$i(0) = I_0 = A + I_N \tag{4.6}$$

From this, we obtain *A* as

$$A = I_0 - I_N \tag{4.7}$$

and the step response of the *RL* circuit becomes

$$i(t) = I_N + (I_0 - I_N)e^{-t/\tau}, \qquad t \geq 0 \tag{4.8}$$

This response is illustrated in Fig. 4-2.

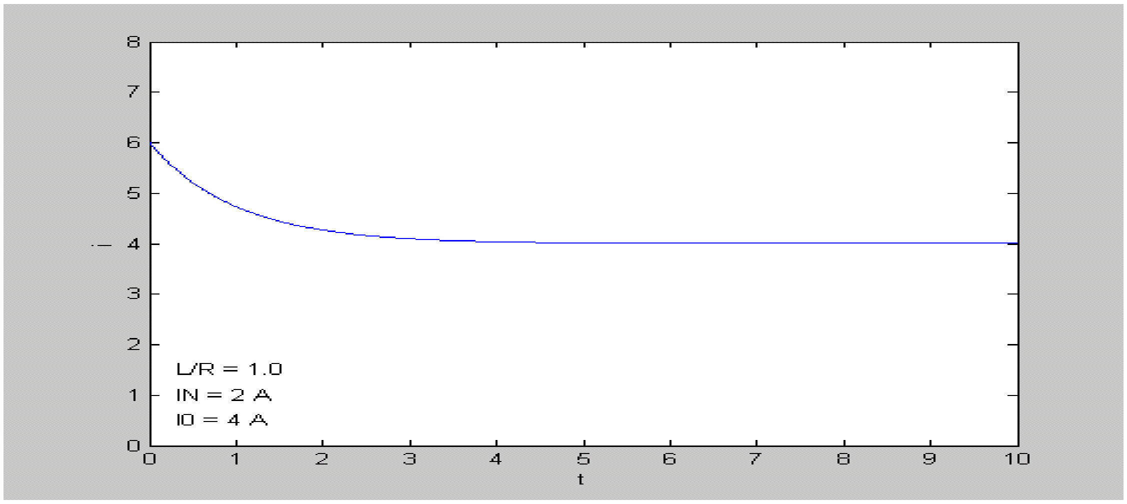


Figure 4-2 Step response of an *RL* circuit

♦

Recall the step responses of the *RC* and *RL* circuits:

$$v(t) = V_T + (V_0 - V_T)e^{-t/\tau}, \quad t \geq 0 \tag{4.9}$$

$$i(t) = I_N + (I_0 - I_N)e^{-t/\tau}, \quad t \geq 0 \tag{4.10}$$

Additional properties of dynamic circuit responses are revealed when these equations are rearranged as

$$v(t) = V_0 e^{-t/\tau} + V_T\left(1 - e^{-t/\tau}\right), \quad t \geq 0 \tag{4.11}$$

$$i(t) = I_0 e^{-t/\tau} + I_N\left(1 - e^{-t/\tau}\right), \quad t \geq 0 \tag{4.12}$$

We recognize the first term on the right in each equation as the zero input response.By definition, the ***zero-input response*** occurs when the input is zero ($V_T = 0$ or $I_N = 0$). The second term on the right in each equation is called the ***zero-state response*** because this part occurs when the initial state of the circuit is zero ($V_0 = 0$ or $I_0 = 0$).

The zero-state response is proportional to the amplitude of the input step function (V_T or I_N). However, the total response (zero input plus zero state) is not directly proportional to the input amplitude.

When the initial state is not zero, the circuit appears to violate the proportionality property of the linear circuits. However, bear in mind that the proportionality property applies to linear circuits with only one input.

The *RC* and *RL* circuits can store energy and have memory. In effect, they have two inputs:

(1) The input that occurred before $t = 0$.
(2) The step function applied at $t = 0$.

The first input produces the initial energy state of the circuit at $t = 0$, and the second causes the zero-state response for $t \geq 0$.

In general, for $t \geq 0$, the total response of a dynamic circuit is the sum of two responses:

(1) the zero-input response caused by the initial conditions produced by inputs applied before $t = 0$, and
(2) the zero-state response caused by the inputs applied after $t = 0$.