Analysis of Golf Ball Motion

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March 23, 2007

Abstract

Projectile motion is a problem addressed in engineering, dynamics and physics courses as well as in real world applications, namely ballistics. There are many potential methods for studying the path of a projectile; four of the most common methods are addressed in this report. The four methods used in analyzing the flight of a golfball were analysis neglecting air friction, analysis assuming constant air friction and two analyses using variable air friction: central differences and Runge-Kutta integration. The results of the flight path of the golf ball were plotted on the same set of axes for comparison. There are obvious differences between the four methods with respect to the predicted path of the golf ball, however because of the way the Runge-Kutta solution is calculated this method produces the most accurate results.

1 Introduction

The purpose of this report is to compare four different methods of analyzing the flight path of a golf ball. This type of dynamics problem is considered simple projectile motion. Often, in order to simplify the calculations, the effects of air resistance on the projectile are neglected.

Neglecting such effects are adequate for simple analyses in order to obtain an approximate trajectory, however there are high precision applications that require such affects be accounted for. Because actual projectiles experience air resistance in the form a drag force, an accurate model of projectile motion must include drag. Including drag forces in the analysis of motion produces a series of differential equations that represent the motion. The differential equations that represent projectile motion are:

$$\frac{dx}{dt} = u \tag{1}$$

$$\frac{dy}{dt} = v \tag{2}$$

$$\frac{du}{dt} = -\frac{C_D Re}{24\tau} u \tag{3}$$

$$\frac{dv}{dt} = -\frac{C_D Re}{24\tau} v - g \tag{4}$$

where u is the velocity of the projectile in the x-direction, v is the velocity of the projectile in the y-direction, C_D is the drag coefficient, Re is the Reynold's number and τ is the time constant.

The system defined by equations 1 through 4 can also be used to analyze the motion if drag is neglected. In this case the first terms in equations 3 and 4 will be equal to zero. Using the above system of equations a comparison of the motion both accounting for and neglecting drag can be obtained.

2

2.1 Derivation of Differential Equations

The first step in the comparison of four analysis methods is to understand how the differential equations 1 through 4 were derived. The following relationships were given in the project description and are necessary for this derivation:

$$D = \frac{1}{2} C_D \rho A V^2 \tag{5}$$

$$Re = \frac{\rho V D}{\mu} \tag{6}$$

$$\tau = \frac{m}{3\pi d \mu} \tag{7}$$

where D is the drag force, ρ is the air density, A is the cross sectional area of the projectile at it's widest point, V is the velocity of the projectile, d is the diameter of the projectile at its widest point and m is the mass of the projectile.

The differential equations can be derived by applying Newton's second law of motion to the free body diagram and kinetic diagram given in Figure 1.

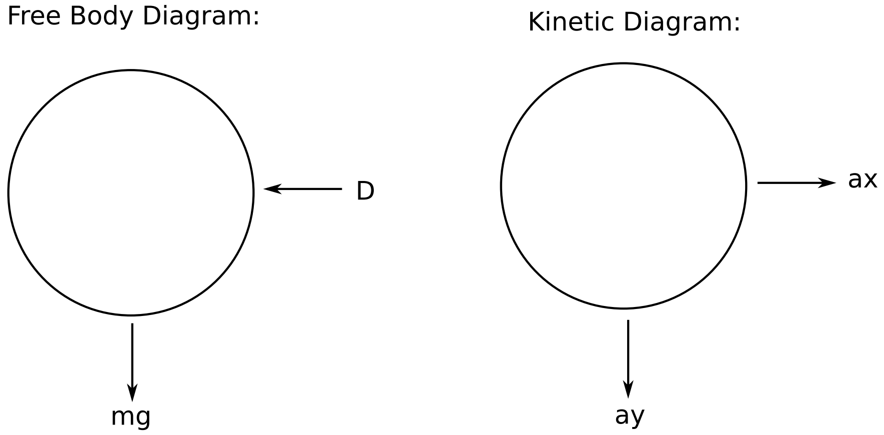

Figure 1: Free Body and Kinetic Diagrams for the Golf Ball in Flight

2.2 Differential Equations in X-Direction

The derivation of the differential equation pertaining to the x-direction is as follows:

$$\sum F_{ext_x} = ma_x$$

$$-D = ma_x$$

From equation 1 we see that

$$a_x = \frac{du}{dt}$$

Using the substituting the above results into the force balance equation gives

$$-\frac{1}{2} C_D \rho A V^2 = m\frac{du}{dt}$$

$$-\frac{C_D \rho A u^2}{2m} = \frac{du}{dt}$$

The cross sectional area of the golf ball, A, is a circle.

$$A = \frac{\pi d^2}{4}$$

$$-\frac{C_D \rho \pi d^2 u^2}{8m} = \frac{du}{dt}$$

Manipulating the given statement for the time constant, τ, gives

$$\frac{1}{3\tau\mu} = \frac{\pi d}{m}$$

$$-\frac{C_D \rho d u^2}{24\tau\mu} = \frac{du}{dt}$$

And finally, substituting the statement for the Reynold's Number into the equation gives the final differential equation for velocity in the x-direction.

$$Re = \frac{\rho d u}{\mu}$$

$$\frac{du}{dt} = -\frac{C_D Re}{24\tau} u$$

2.3 Differential Equations in Y-Direction

The derivation of the equations in the y-direction are identical to those in the x-direction, except that the force of gravity, mg, acts in the y-direction.

$$-D - mg = \sum m a_y$$

$$a_y = \frac{dv}{dt}$$

$$\frac{-D - mg}{m} = \frac{dv}{dt}$$

$$-\frac{C_D \rho A v^2}{2m} - g = \frac{dv}{dt}$$

$$A = \frac{\pi d^2}{4}$$

$$-\frac{C_D \rho \pi d^2 v^2}{8m} - g = \frac{dv}{dt}$$

$$\frac{1}{3\tau\mu} = \frac{\pi d}{m}$$

$$-\frac{C_D \rho d v^2}{24\tau\mu} - g = \frac{dv}{dt}$$

$$Re = \frac{\rho d v}{\mu}$$

$$\frac{dv}{dt} = -\frac{C_D Re}{24\tau} v - g$$

2.4 Analysis of Motion

The following information was given for the flight of the golf ball:

Weight (mg)	$1.5oz$
Diameter (d)	$1.75in$
Air Viscosity (μ)	$0.375 \times 10^{-6} \frac{lb-sec}{ft}$
Initial Velocity (V_0)	$120 \frac{ft}{sec}$
Initial Angle (θ)	30° from horizontal
Acceleration due to gravity (g)	$32.2 \frac{ft}{sec^2}$

2.4.1 Neglecting Friction

The simplest method of analyzing the flight of the golf ball is to neglect the effects of the drag force. This was the first method used in the comparison of analysis techniques.

Assuming no air friction on the golf ball, the accelerations in the x and y-directions become:

$$a_x = 0$$

$$a_y = -g$$

From kinematics, the equation for distance, x, as a function of time is

$$x(t) = x_0 + V_{0x}t$$

where x_0 is the projectile's initial position and V_{0x} is the initial velocity in the x-direction. V_{0x} can be expressed in terms of the initial velocity, V_0, and the initial launch angle, θ:

$$V_{0x} = V_0 cos\theta$$

$$V_{0x} = 103.92\frac{ft}{sec}$$

When the initial position of the golf ball is assumed to be located at the origin, $(0, 0)$, the distance of the ball as a function of time becomes:

$$x(t) = 0 + (V_0 cos\theta)t$$

The equation for the height, y, of the ball in terms of time is:

$$y(t) = y_0 + V_{0y}t - \frac{1}{2}gt^2$$

where y_0 is the initial height and V_{0y} is the initial velocity in the y-direction. Again, V_{0y} can be expressed in terms if the initial velocity, V_0, and the launch angle, θ.

$$V_{0y} = V_0 sin\theta$$

$$V_{0y} = 60\frac{ft}{sec}$$

If the starting position is again assumed to be the origin, the height of the ball with respect to time is given as:

$$y(t) = 0 + (V_0 sin\theta)t - \frac{1}{2}gt^2$$

Using equations from kinematics and projectile motion, the equation for the maximum range of the golf ball is

$$x_{max} = \frac{V_0^2 sin2\theta}{g}$$

$$x_{max} = 387.29ft$$

And the equation for the maximum height of the golf ball is

$$y_{max} = \frac{(V_0 sin\theta)^2}{2g}$$

$$y_{max} = 55.90ft$$

The equations for the velocities in the x and y-directions can be found by taking the time derivatives of the equation for distance, x, and height, y:

$$\frac{dx}{dt} = u$$

$$\frac{dy}{dt} = v$$

$$u(t) = V_0 cos\theta$$

$$v(t) = V_0 sin\theta - gt$$

These position equations were plotted in Figure 2 for comparison with the other analysis methods.

2.4.2 Constant Air Friction

The analysis with constant drag force will produce more accurate results that the analysis with no drag. The differential equations, equations 3 and 4, were used in the analysis with constant drag.

To simplify the math in this model, the time derivatives of the velocities in the x ($\frac{du}{dt}$) and y ($\frac{dv}{dt}$) directions will be restated as $\dot{u}$ and $\dot{v}$. The differential equation for the velocity in the x-direction then becomes:

$$\dot{u} = -\frac{C_D Re}{24\tau} u \tag{8}$$

This is a homogeneous differential equation. The solution will be of the form

$$u_h = \alpha e^{\beta t}$$

And the time derivative of the solution will be

$$\dot{u}_h = \alpha\beta e^{\beta t}$$

Substituting the solution into equation 8 gives

$$\alpha\beta e^{\beta t} + \left(\frac{C_D Re}{24\tau}\right)\alpha e^{\beta t} = 0$$

which was then solved for β:

$$\beta + \left(\frac{C_D Re}{24\tau}\right) = 0$$

$$\beta = -\left(\frac{C_D Re}{24\tau}\right)$$

The general solution of the differential equation then becomes

$$u_h = \alpha e^{-\left(\frac{C_D Re}{24\tau}\right)t}$$

The given initial conditions were then used to solve for α.

$$u(0) = V_0 cos\theta$$

$$V_0 cos\theta = \alpha e^{-\left(\frac{C_D Re}{24\tau}\right)(0)}$$

$$\alpha = V_0 cos\theta$$

The velocity in the x-direction, u, as a function of time is:

$$u(t) = (V_0 cos\theta) e^{-\left(\frac{C_D Re}{24\tau}\right)t}$$

The distance, x, was then found by integrating the above equation between time 0 and time t.

$$u(t) = \frac{dx}{dt} = (V_0 cos\theta) e^{-\left(\frac{C_D Re}{24\tau}\right)t}$$

$$dx = (V_0 cos\theta) e^{-\left(\frac{C_D Re}{24\tau}\right)t} dt$$

$$\int_0^t dx = \int_0^t (V_0 cos\theta) e^{-\left(\frac{C_D Re}{24\tau}\right)t} dt$$

$$x(t) = -\frac{24\tau V_0 cos\theta}{C_D Re} e^{-\frac{C_D Re}{24\tau}t} + \frac{24\tau V_0 cos\theta}{C_D Re}$$

The same basic procedure was used to solve for the height, y, as a function of time. In the y-direction, however, the force of gravity is included in the differential equation making it non-homogeneous.

$$\dot{v} + \frac{C_D Re}{24\tau} v = g$$

$$v = \alpha e^{\beta t}$$

$$\dot{v} = \alpha \beta e^{\beta t}$$

The first step in the solution for the y-direction was to find the homogeneous solution.

$$\alpha \beta e^{\beta t} + \frac{C_D Re}{24\tau} \alpha e^{\beta t} = 0$$

$$\beta = -\frac{C_D Re}{24\tau}$$

$$v_h = \alpha e^{-\frac{C_D Re}{24\tau}t}$$

Because the differential equation equals the acceleration due to gravity, g, which is constant the particular solution will be a constant.

$$v_p = A$$

$$\dot{v}_p = 0$$

$$0 + A\Big(\frac{C_D Re}{24\tau}\Big) = -g$$

$$A = -\frac{24\tau g}{C_D Re}$$

$$v_p(t) = -\frac{24\tau g}{C_D Re}$$

Adding the homogeneous and particular solutions yielded the solution for the velocity in the y-direction.

$$v(t) = \alpha e^{-\frac{C_D Re}{24\tau}t} - \frac{24\tau g}{C_D Re}$$

Again, the initial condition $v(0) = V_0 cos\theta$ was used to calculate α and the result was integrated to find the height, y, as a function of time.

$$y(t) = \Big(V_0 sin\theta + \frac{24\tau g}{C_D Re}\Big)\Big(-\frac{24\tau}{C_D Re}e^{-\frac{C_D Re}{24\tau}t}\Big) + \Big(\Big(V_0 sin\theta + \frac{24\tau}{C_D Re}\Big)\Big(\frac{24\tau}{C_D Re}\Big)\Big) - \Big(\frac{24\tau}{C_D Re}\Big)t$$

The position equations were then added to the plot in Figure 2 for comparison with the other methods.

2.4.3 Method of Central Differences

Adding variable air friction to the model adds a degree of difficulty to the calculations. The drag force is dependent on the velocity of the golf ball at a given time, therefore the drag force changes as the ball's velocity changes. The best way to calculate the flight of the golf ball with variable drag force is through a numerical approach using computer software. The first numerical approach used in this problem was the Method of Central Differences. The Method of Central Differences manipulates the differential equation to find the velocities over a small time step. With velocities known over a given time step, the position over the same time step can be calculated.

The following four equations were given for the implementation of the Method of Central Differences:

$$x(t) = x(t - \Delta t) + \frac{\Delta t}{2}\Big[u(t - \Delta t) + u(t)\Big]$$

$$y(t) = y(t - \Delta t) + \frac{\Delta t}{2}\Big[v(t - \Delta t) + v(t)\Big]$$

$$u(t) = u(t - \Delta t) - \frac{C_D Re}{24\tau}\left[\frac{u(t) + u(t - \Delta t)}{2}\right]\Delta t$$

$$v(t) = v(t - \Delta t) - \frac{C_D Re}{24\tau}\left[\frac{v(t) + v(t - \Delta t)}{2}\right]\Delta t - g\Delta t$$

When using the Method of Central Differences, the time t is the start of any given time step and Δt is the width of the step. Therefore, $t - \Delta t$ is the initial time, t_0. The following relationships were derived using this characteristic of the time step.

$$x(t - \Delta t) = x_0$$

$$y(t - \Delta t) = y_0$$

$$u(t - \Delta t) = u_0$$

$$v(t - \Delta t) = v_0$$

Substituting these relationships into the given equations gives:

$$x(t) = x_0 + \frac{\Delta t}{2}\Big[u_0 + u(t)\Big]$$

$$y(t) = y_0 + \frac{\Delta t}{2}\Big[v_0 + v(t)\Big]$$

$$u(t) = u_0 - \frac{C_D Re}{24\tau}\left[\frac{u(t) + u_0}{2}\right]\Delta t$$

$$v(t) = u_0 - \frac{C_D Re}{24\tau}\left[\frac{v(t) + v_0}{2}\right]\Delta t - g\Delta t$$

The above equations for the velocities were then rearranged to isolate the $u(t)$ and $v(t)$ terms on the left side of the equations. The final differential equations used for the Method of Central Differences were:

$$u(t) = u_0\left[\frac{48\tau - C_D Re \Delta t}{48\tau + C_D Re \Delta t}\right]$$

and

$$v(t) = v_0\left[\frac{48\tau - C_D Re \Delta t}{48\tau + C_D Re \Delta t}\right] - \left[\frac{48 d\tau \Delta t}{48\tau + C_D Re \Delta t}\right]$$

These equations were input into a MatLab program designed to iterate through the small time steps. The output of the MatLab program was added to the plot in Figure 2 in order to compare the four analysis methods.

2.4.4 Runge-Kutta Integration

Runge-Kutta integration is the second method used to calculate the golf ball's trajectory with variable drag force. The Runge-Kutta method uses differential equations 1 through 4 directly without the need for further manipulation. This method iterates through the steps of the equations much like the Method of Central Differences but with a smaller degree of error. Runge-Kutta achieves greater accuracy by also testing a value at the midpoint of an interval in order to eliminate errors caused by large steps. The differential equations of the system were input into a MatLab program and the results of the Runge-Kutta integration were added to the plot in Figure 2 as the final method for comparison.

The MatLab program used to create the plot in Figure 2 is given in Appendix A.

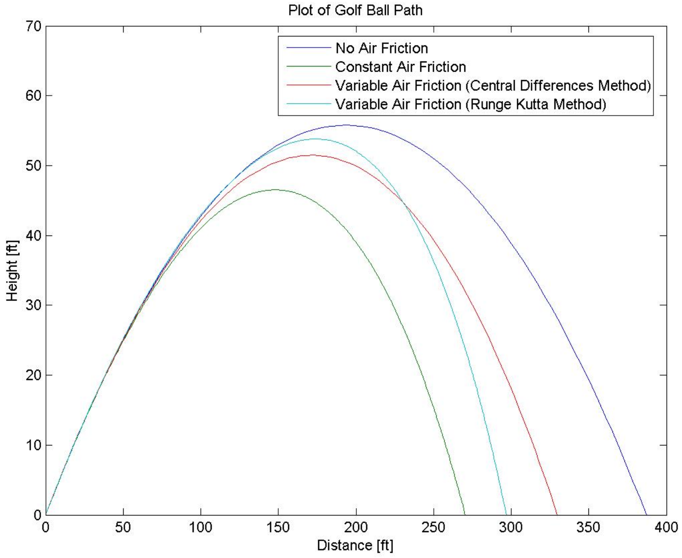

Figure 2: Comparison Plot of 4 Flight Analysis Methods

3 Discussion

Figure 2 shows the results of the four different methods that were used to analyze the flight of the golf ball. The first analysis was done assuming that there was no drag (friction due to the air) acting on the ball. Neglecting drag removes the only force that has the potential to slow the golf ball down. This caused the trajectory of the golf ball with this method to have the greatest maximum height and range of all four methods. Without drag the golf ball will reach a maximum height of 55.9 ft and a range of 387 ft.

The second method was analysis assuming that the drag acting on the golf ball was constant. The results of this analysis in Figure 2 show that this assumption produces the trajectory with the lowest height and the shortest range. This is because this method assumes that the product of C_D and Re remains constant. Without the assumption of constant drag, the Reynold's number is directly dependent on the ball's velocity and will therefore be changing in time. As drag slows the ball down, the Reynold's number will also be smaller. By assuming constant drag the drag force will remain large at lower velocities where it should decrease. This increases the overall effect the drag has on the flight of the ball and greatly reduces both range and maximum height.

The two numerical methods examined produced the most accurate results for the trajectory of the golf ball. The first, the Method of Central Differences, produced a path that was lower and shorter than the no-drag model but longer and higher than the constant drag model. This is expected as the Method of Central Differences takes the changes in the Reynold's number into account. This decreases the drag force at lower velocities which leads to an overall increase in height and range over the constant drag model. The Method of Central Differences has some inherent error because of the way it is calculated; it relies on approximating the velocity and position at a given time based on the velocity and position of the previous time. This method has a tendency to introduce error into the calculations.

The second numerical method, Runge-Kutta integration, does not rely on approximate values. All of the differential equations used in Runge-Kutta, equations 1 through 4, rely only on themselves. As the values of one equation changes it forces a change in the others. The Method of Central Differences cannot take this instantaneous variation into account. Runge-Kutta, however, calculates the values of all the differential equations at the same time. This is why the Runge-Kutta method produces a path with a higher maximum height than the Method of Central Differences but lower than the no-friction model. Despite having a greater maximum height, the Runge-Kutta model has a shorter range than the Method of Central Differences.

4 Conclusions

This project was meant to compare four different methods of analyzing projectile motion. The projectile in question was a simple golf ball. The four models used to analyze the motion were neglecting air friction, constant air friction and variable air friction using both the Method of Central Differences and Runge-Kutta integration. MatLab was used to calculate and plot the results of all four methods. The MatLab program is given in Appendix A and the plot is given in Figure 2. The trajectories found via the four models are all significantly different. Based on the methodology used in each of the four analyses, it can be concluded that Runge-Kutta integration produces the most accurate results for projectile motion.

A MatLab Program

```
 function ydot = frict(t,y)
g = 32.2;
w = 1.5/16; %lbs
m = w/g; %slugs
mu = 0.375 * 10^(-6); %(lb-sec)/ft
d = 1.75 / 12; %ft
roe = .002378; %slugs/ft3
tau = m / (3*pi*d*mu);
Reu = (roe * y(3) * d)/ mu;
Rev = (roe * y(4) * d)/ mu;
if (Reu <= 9*104)
Cd = 0.4;
else
Cd = 0.1;
end
ydot = [y(3);y(4);-Cd*Reu/(24*tau)*y(3); -Cd*Rev/(24*tau)*y(4)-g];
end


% Finite Difference Approach to Solve ODE System
% ****Initializing Vectors and Specifying Constants****
g = 32.2;
w = 1.5/16; %lbs
m = w/g; %slugs
Vo = 120; %ft/sec
angle = pi / 6; %radians
i=1;
x = [];
y = [];
u = [];
v = [];
mu = 0.375 * 10^(-6); %(lb-sec)/ft
d = 1.75 / 12; %ft
roe = .002378; %slugs/ft3
tau = m / (3*pi*d*mu);
x(i) = 0;
y(i) = 0.001;
u(i) = Vo * sqrt(3)/2;
v(i) = Vo * .5;
dt = 0.04;
t = [];
t(i) = 0;
s = t(1);


% no air friction
```

```
p = 0:4.4:400;
for j=1:91
q(j) = p(j)*tan(.523) - .5*32.2 / 1202 * p(j)2 * (1 + (tan(.523))2);
end


% constant air friction
j=1;
a=[];
z=[];
a(j)=0;
z(j)=0.001;
n=0;
while z(j) > eps
n=n+.01;
j=j+1;
a(j) = -24*tau*Vo*sqrt(3)/2/22500*exp(-22500/(24*tau)*n) + 24*tau*Vo*
sqrt(3)/2/22500;
z(j) = (.5*Vo+24*tau*g/22500)*(-24*tau/22500*exp(-22500/(24*tau)*n)) +
((Vo*.5 + 24*tau*g/22500)*(24*tau/22500))-(24*g*tau/22500*n);
end


% method of central differences
while y(i) > eps
i = i+1;
Reu = (roe * u * d)/ mu;
Rev = (roe * v * d)/ mu;
if (Reu <= 9*104)
Cd = 0.4;
else
Cd = 0.1;
end
u(i)=(u(i-1)*(1 - Cd*Reu*dt/(48*tau))) / (1 + Cd*Reu*dt/(48*tau));
v(i)=(v(i-1)*(1 - Cd*Reu*dt/(48*tau)) - g*dt) / (1 + Cd*Reu*dt/(48*tau));
x(i)=x(i-1) + dt/2 * (u(i-1) + u(i));
y(i)=y(i-1) + dt/2 * (v(i-1) + v(i));
t(i) = s + (i-1)*dt;
end
velocity = sqrt( u(i)2 + v(i)2 );
degrees = atan(v(i)/u(i)) * 180 / pi;
fprintf('Time = %-5.2f sec \n',t(i));
fprintf('Distance = %-5.4f feet \n',x(i));
fprintf('Velocity = %-5.4f feet/second at %-5.4f degrees\n', velocity,
degrees);
```

```
% Runge Kutta Method
Vo = 120; %ft/sec
u0 = Vo * sqrt(3)/2;
v0 = Vo * .5;
x0=0;
y0=0;
z0 = [x0;y0;u0;v0];
for j = 1:600
tspan(j) = 0 + (j-1)*.01;
end
[t,m] = ode45(@frict,tspan,z0);


% plot results
plot(p,q,a,z,x,y,m(:,1),m(:,2));
title('Plot of Golf Ball Path');
xlabel('Distance [ft]');
ylabel('Height [ft]');
axis([0,400,0,70]);
legend('No Air Friction','Constant Air Friction','Variable Air Friction (Central
Differences Method)','Variable Air Friction (Runge Kutta Method)');
```