2. TRANSLATION

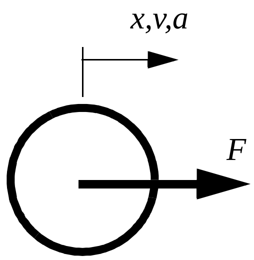
Topics:
- Basic laws of motion
- Gravity, inertia, springs, dampers, cables and pulleys, drag, friction, FBDs
- System analysis techniques
- Design case

Objectives:
- To be able to develop differential equations that describe translating systems.

2.1 INTRODUCTION

If the velocity and acceleration of a body are both zero then the body will be static. If the applied forces are balanced, and cancel each other out, the body will not accelerate. If the forces are unbalanced then the body will accelerate. If all of the forces act through the center of mass then the body will only translate. Forces that do not act through the center of mass will also cause rotation to occur. This chapter will focus only on translational systems.

The equations of motion for translating bodies are shown in Figure 2.1. These state simply that velocity is the first derivative of position, and velocity is the first derivative of acceleration. Conversely the acceleration can be integrated to find velocity, and the velocity can be integrated to find position. Therefore, if we know the acceleration of a body, we can determine the velocity and position. Finally, when a force is applied to a mass, an acceleration can be found by dividing the net force by the mass.



equations of motion

$$v(t) = \left(\frac{d}{dt}\right)x(t) \qquad (1)$$

$$a(t) = \left(\frac{d}{dt}\right)^2 x(t) = \left(\frac{d}{dt}\right)v(t) \qquad (2)$$

OR

$$x(t) = \int v(t)dt = \int\int a(t)dt \qquad (3)$$

$$v(t) = \int a(t)dt \qquad (4)$$

$$a(t) = \frac{F(t)}{M} \qquad (5)$$

where,

x, v, a = *position, velocity and acceleration*

M = *mass of the body*

F = *an applied force*

Figure 2.1 Velocity and acceleration of a translating mass

An example application of these fundamental laws is shown in Figure 2.2. The initial conditions of the system are supplied (and are normally required to solve this type of problem). These are then used to find the state of the system after a period of time. The solution begins by integrating the acceleration, and using the initial velocity value for the integration constant. So at t=0 the velocity will be equal to the initial velocity. This is then integrated once more to provide the position of the object. As before, the initial position is used for the integration constant. This equation is then used to calculate the position after a period of time. Notice that the units are used throughout the calculations, this is a good practice for any engineer.

Given an initial (t=0) state of x=5m, v=2m/s, a=3ms^{-2}, find the system state 5 seconds later assuming constant acceleration.

The initial conditions for the system at time t=0 are,

$$x_0 = 5m$$
$$v_0 = 2ms^{-1}$$
$$a_0 = 3ms^{-2}$$

Note: units are very important and should normally be carried through all calculations.

The constant acceleration can be integrated to find the velocity as a function of time.

$$v(t) = \int a_0 dt = a_0 t + C = a_0 t + v_0 \qquad (6)$$

Note:

$$v(t) = a_0 t + C$$
$$v_0 = a_0(0) + C$$
$$v_0 = C$$

Next, the velocity can be integrated to find the position as a function of time.

$$x(t) = \int v(t)dt = \int (a_0 t + v_0)dt = \frac{a_0}{2}t^2 + v_0 t + x_0 \qquad (7)$$

This can then be used to calculate the position of the mass after 5 seconds.

$$x(5) = \frac{a_0}{2}t^2 + v_0 t + x_0$$
$$= \frac{3ms^{-2}}{2}(5s)^2 + 2ms^{-1}(5s) + 5m$$
$$= 37.5m + 10m + 5m = 52.5m$$

Figure 2.2 Sample calculation for a translating mass, with initial conditions

2.2 MODELING

When modeling translational systems it is common to break the system into parts. These parts are then described with Free Body Diagrams (FBDs). Common components that must be considered when constructing FBDs are listed below, and are discussed in following sections.

- gravity and other fields - apply non-contact forces
- inertia - opposes acceleration and deceleration
- springs - resist deflection
- dampers and drag - resist motion
- friction - opposes relative motion between bodies in contact

- cables and pulleys - redirect forces
- contact points/joints - transmit forces through up to 3 degrees of freedom

2.2.1 Free Body Diagrams

Free Body Diagrams (FBDs) allow us to reduce a complex mechanical system into smaller, more manageable pieces. The forces applied to the FBD can then be summed to provide an equation for the piece. These equations can then be used later to do an analysis of system behavior. These are required elements for any engineering problem involving rigid bodies.

An example of FBD construction is shown in Figure 2.3. In this case there is a mass sitting atop a spring. An FBD can be drawn for the mass. In total there are two obvious forces applied to the mass, gravity pulling the mass downward, and a spring pushing the mass upwards. The FBD for the spring has two forces applied at either end. Notice that the spring force, FR1, acting on the mass, and on the spring have an equal magnitude, but opposite direction.

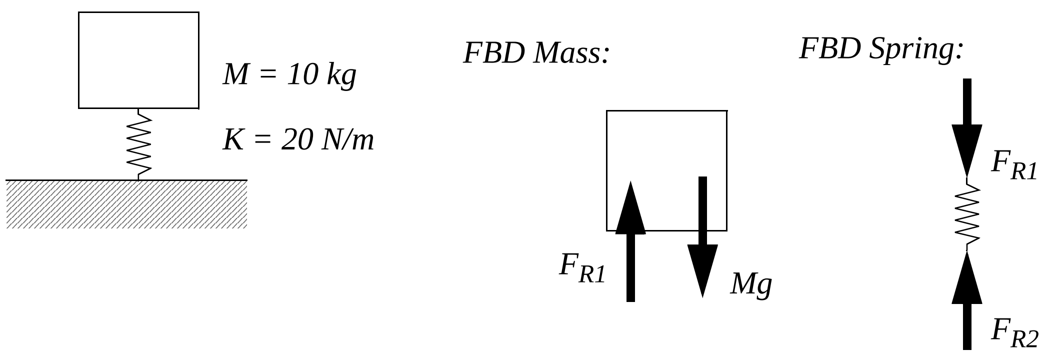


Figure 2.3 Free body diagram example

2.2.2 Mass and Inertia

In a static system the sum of forces is zero and nothing is in motion. In a dynamic system the sum of forces is not zero and the masses accelerate. The resulting imbalance in forces acts on the mass causing it to accelerate. For the purposes of calculation we create a virtual reaction force, called the inertial force. This force is also known as D'Alembert's (pronounced as daa-lamb-bears) force. It can be included in calculations in one of two ways. The first is to add the inertial force to the FBD and then add it into the sum of forces, which will equal zero. The second method is known as D'Alembert's equation where all of the forces are summed and set equal to the inertial force, as shown in Figure 2.4. The acceleration is proportional to the inertial force and inversely proportional to the mass.

$$\sum F = Ma \qquad \textit{(Newton's)} \qquad (11)$$

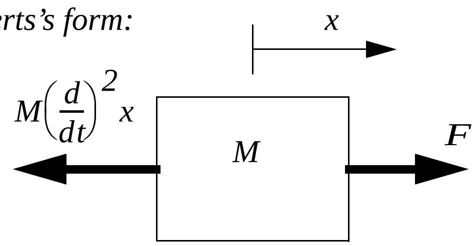
$$\sum F - Ma = 0 \qquad \textit{(D'Alembert's)} \qquad (12)$$

Figure 2.4 D'Alembert's and Newton's equations

An application of Newton's equation to FBDs can be seen in Figure 2.5. In the first case an inertial force is added to the FBD. This force should be in an opposite direction (left here) to the positive direction of the mass (right). When the sum of forces equation is used then the force is added in as a normal force component. In the second case Newton's equation is used so the force is left off the FBD, but added to the final equation. In this case the sign of the inertial force is positive if the assumed positive direction of the mass matches the positive direction for the summation.

D'Alemberts's form:

$$Ma = M\left(\frac{d}{dt}\right)^2 x$$



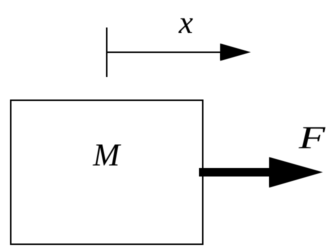
$$\overset{+}{\rightarrow}\sum F_x = F - M\left(\frac{d}{dt}\right)^2 x = 0$$

or

$$\overset{+}{\leftarrow}\sum F_x = -F + M\left(\frac{d}{dt}\right)^2 x = 0$$

Note: If using an inertial force then the direction of the force should be opposite to the positive motion direction for the mass.

Newton's form:



$$\overset{+}{\rightarrow}\sum F_x = F = M\left(\frac{d}{dt}\right)^2 x$$

or

$$\overset{+}{\leftarrow}\sum F_x = -F = -M\left(\frac{d}{dt}\right)^2 x$$

Note: If using Newton's form the sign of the inertial force should be positive if the positive direction for the summation and the mass are the same, otherwise if they are opposite then the sign should be negative.

Figure 2.5 Free body diagram and inertial forces

An example of the application of Newton's equation is shown in Figure 2.6. In this example there are two unbalanced forces applied to a mass. These forces are summed and set equal to the inertial force. Solving the resulting equation results in acceleration values in the 'x' and 'y' directions. In this example the forces and calculations are done in vector form for convenience and brevity.

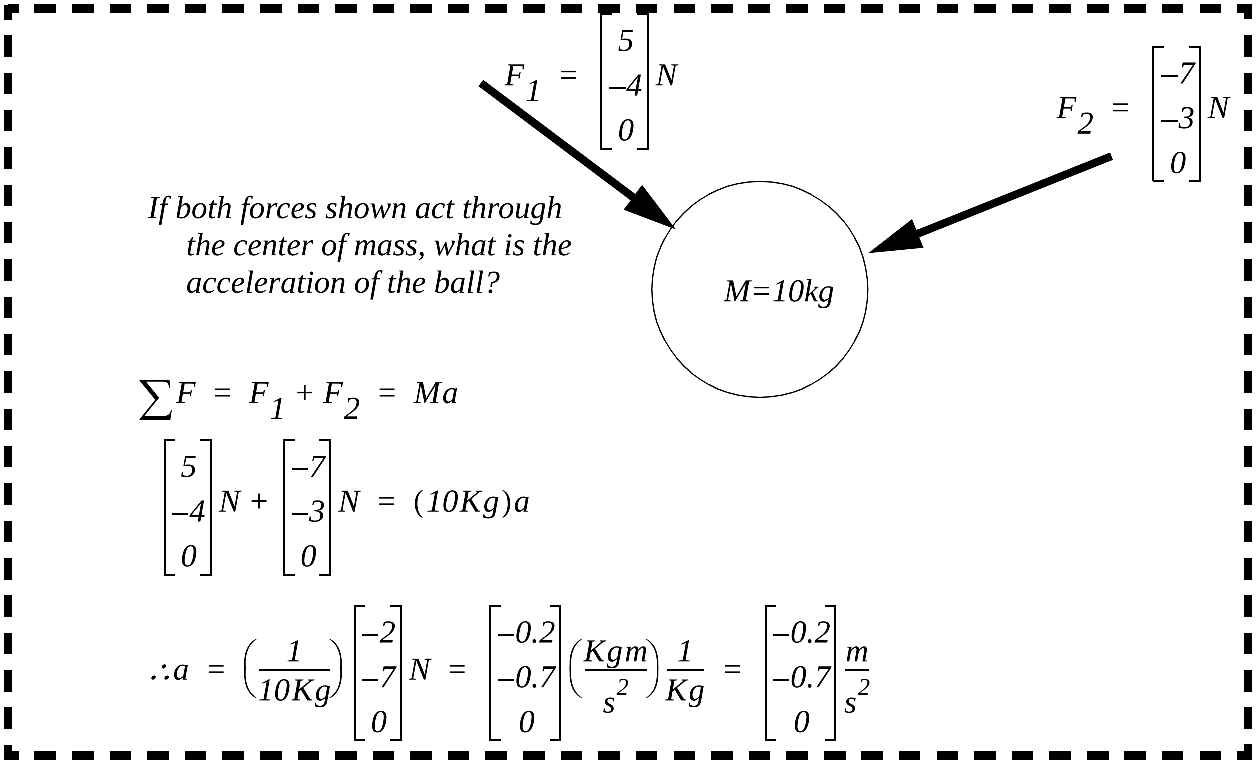


Figure 2.6 Sample acceleration calculation

```
// A program to sum forces and calculate the acceleration

// define the given forces and mass
F1 = [5, -4, 0];
F2 = [-7, -3, 0];
M = 10;

function foo=Sum() // The sum of the applied forces
        foo = F1 + F2;
endfunction

A = Sum() / M;
printf("The acceleration is ( %f, %f, %f) m/s^2 \n", A(1), A(2), A(3));
printf("The magnitude is |A| = %f m/s^2 \n", norm(A));
```

Figure 2.7 A Scilab calculation example

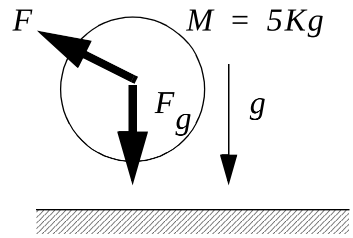
2.2.3 Gravity and Other Fields

Gravity is a weak force of attraction between masses. In our situation we are in the proximity of a large mass (the earth) which produces a large force of attraction. When analyzing forces acting on rigid bodies we add this force to our FBDs. The magnitude of the force is proportional to the mass of the object, with a direction toward the center of the earth (down).

The relationship between mass and force is clear in the metric system with mass having the units Kilograms (kg), and force the units Newtons (N). The relationship between these is the gravitational constant 9.81N/kg, which will vary slightly over the surface of the earth. The Imperial units for force and mass are both the pound (lb.) which often causes confusion. To reduce this confusion the unit for force is normally modified to be, lbf.

An example calculation including gravitational acceleration is shown in Figure 2.8. The 5kg mass is pulled by two forces, gravity and the arbitrary force ’F’. These forces are described in vector form, with the positive ’z’ axis pointing upwards. To find the equations of motion the forces are summed. To eliminate the second derivative on the inertia term the equation is integrated twice. The result is a set of three vector equations that describe the x, y and z components of the motion. Notice that the units have been carried through these calculations.

Assume we have a mass that is acted upon by gravity and a second constant force vector. To find the position of the mass as a function of time we first define the gravity vector and position components for the system.



$$g = \begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{N}{Kg} = \begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} \qquad X(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \qquad F = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \qquad F_g = Mg$$

Next, sum the forces and set them equal to inertial resistance.

$$\sum F = Mg + F = M\left(\frac{d}{dt}\right)^2 X(t)$$

$$5Kg \begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} + \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = 5Kg\left(\frac{d}{dt}\right)^2 \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} + 0.2Kg^{-1} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \left(\frac{d}{dt}\right)^2 \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Integrate twice to find the position components.

$$\left[\begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} + 0.2Kg^{-1} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \right] t + \begin{bmatrix} v_{x_0} \\ v_{y_0} \\ v_{z_0} \end{bmatrix} = \left(\frac{d}{dt}\right) \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\frac{1}{2}\left[\begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} + 0.2Kg^{-1} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \right] t^2 + \begin{bmatrix} v_{x_0} \\ v_{y_0} \\ v_{z_0} \end{bmatrix} t + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 0.1 f_x t^2 + v_{x_0} t + x_0 \\ 0.1 f_y t^2 + v_{y_0} t + y_0 \\ \left(\frac{-9.81}{2} + 0.1 f_z\right) t^2 + v_{z_0} t + z_0 \end{bmatrix} m$$

Note: When an engineer solves a problem they will always be looking at the equations and unknowns. In this case there are three equations, and there are 9 constants/givens fx, fy, fz, vx0, vy0, xz0, x0, y0 and z0. There are 4 variables/unknowns x, y, z and t. Therefore with 3 equations and 4 unknowns only one value (4-3) is required to find all of the unknown values.

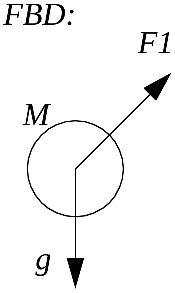
Figure 2.8 Gravity vector calculations

Like gravity, magnetic and electrostatic fields can also apply forces to objects. Magnetic forces are commonly found in motors and other electrical actuators. Electrostatic forces are less common, but may need to be considered for highly charged systems.

Given,

$$F_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} N \qquad g = \begin{bmatrix} 0 \\ -9.81 \\ 0 \end{bmatrix} \frac{N}{Kg} \qquad M = 2Kg$$

FBD:



Find the acceleration.

ans.

$$a = \begin{bmatrix} 1.5 \\ -7.81 \\ 0 \end{bmatrix} \frac{m}{s}$$

Figure 2.9 Drill problem: find the acceleration of the FBD

2.2.4 Springs

Springs are typically constructed with elastic materials such as metals and plastics, that will provide an opposing force when deformed. The properties of the spring are determined by the Young's modulus (E) of the material and the geometry of the spring. A primitive spring is shown in Figure 2.10. In this case the spring is a solid member. The relationship between force and displacement is determined by the basic mechanics of materials relationship. In practice springs are more complex, but the parameters (E, A and L) are combined into a more convenient form. This form is known as Hooke's Law.

$$\delta = \left(\frac{L}{EA}\right)F$$

$$F = \left(\frac{EA}{L}\right)\delta$$

$$F = K\delta \qquad \text{(Hooke's law)}$$

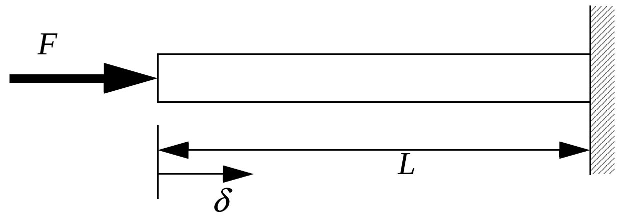
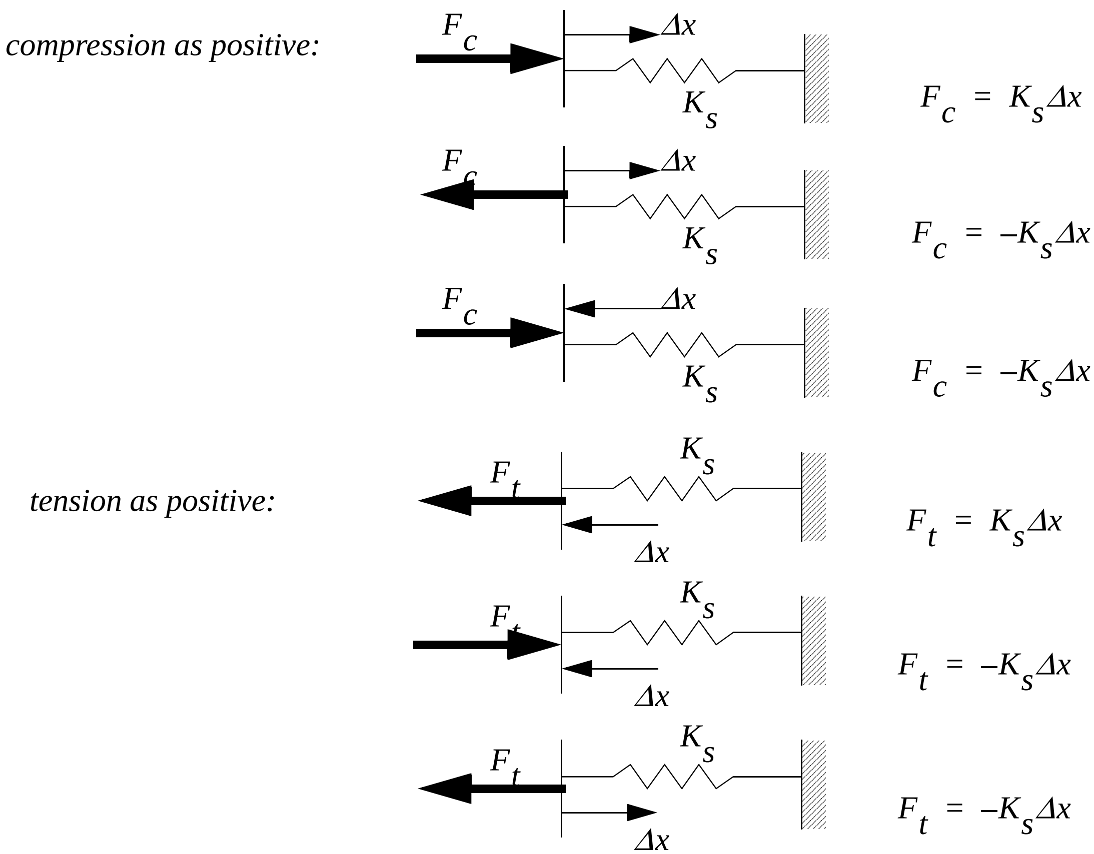


Figure 2.10 A solid member as a spring

Hooke’s law does have some limitations that engineers must consider. The basic equation is linear, but as a spring is deformed the material approaches plastic deformation, and the modulus of elasticity will change. In addition the geometry of the object changes, also changing the effective stiffness. Springs are normally assumed to be massless. This allows the inertial effects to be ignored, such as a force propagation delay. In applications with fast rates of change the spring mass may become significant, and they will no longer act as an ideal device.

The cases for tension and compression are shown in Figure 2.11. In the case of compression the spring length has been made shorter than its’ normal length. This requires that a compression force be applied. For tension, both the displacement from neutral and the required force reverse direction. It is advisable when solving problems to assume a spring is either in tension or compression, and then select the displacement and force directions accordingly.

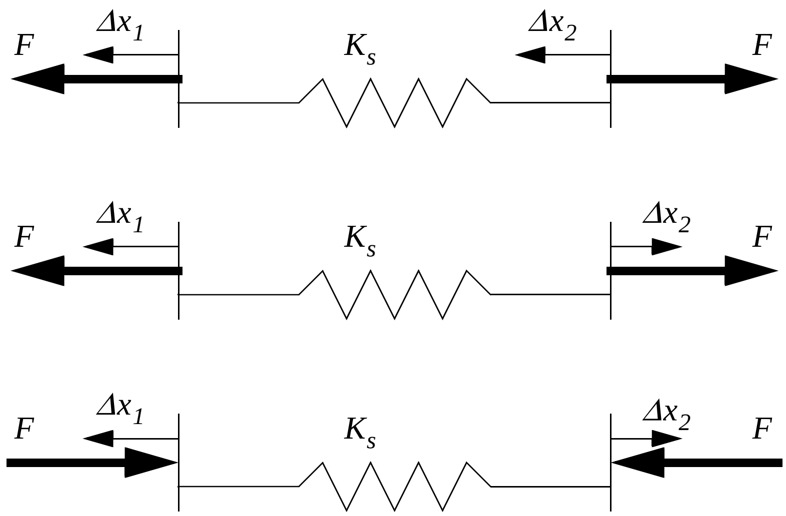
Δx = deformed length *ASIDE: a spring has a natural or undeformed length. When at this length it is neither in tension or compression*



NOTE: the symbols for springs, resistors and inductors are quite often the same or similar. You will need to remember this when dealing with complex systems - and especially in this book where we deal with both types of components.

Figure 2.11 Sign conventions for spring forces and displacements

Previous examples have shown springs with displacements at one end only. In Figure 2.12, springs are shown that have movement at both ends. In these cases the sign of the force applied to the spring is selected with reference to the assumed compression or tension. The primary difference is that care is required to correctly construct the expressions for the tension or compression forces. In all cases the forces on the springs must be assumed and drawn as either tensile or compressive. In the first example the displacement and forces are tensile. The displacement at the left is tensile, so it will be positive, but on the right hand side the displacement is compressive so it is negative. In the second example the force and both displacements are shown as tensile, so the terms are both positive. In the third example the force is shown as compressive, while the displacements are both shown as tensile, so both terms are negative.



$$F = K_s(\Delta x_1 - \Delta x_2)$$

$$F = K_s(\Delta x_1 + \Delta x_2)$$

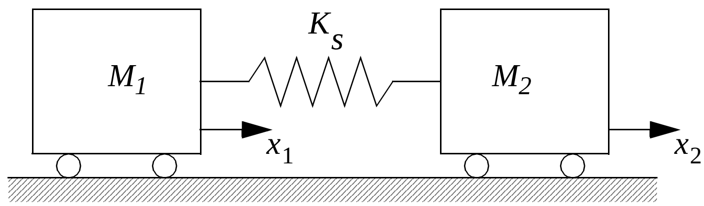
$$F = K_s(-\Delta x_1 - \Delta x_2)$$

Aside: it is useful to assume that the spring is either in tension or compression, and then make all decisions based on that assumption.

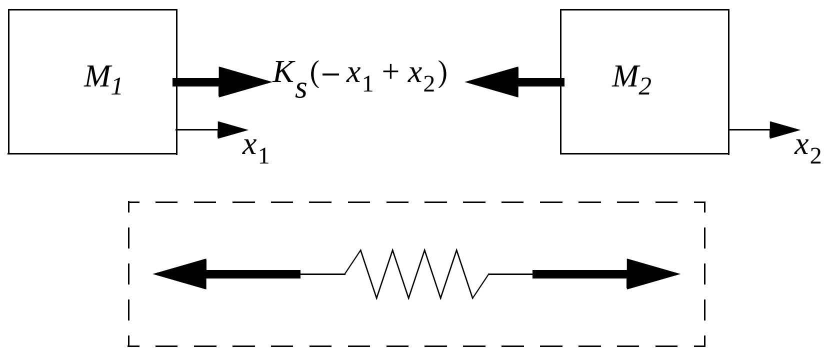
Figure 2.12 Examples of forces when both sides of a spring can move

The example in Figure 2.13 shows two masses separated by a spring. In the first example the spring is assumed to be in tension. When x1 becomes positive it will put the spring in compression, so it is made negative, however a positive x2 will put the spring in tension, so it remains positive. In the second case the spring is assumed to be in compression and this time x2 will put it in tension so it is made negative.

Consider the two masses below separated by a spring.



The system can be reduced to free body diagrams assuming the spring is tension.



Note: In this example the spring is assumed to be in tension and the signs of the magnitude are made negative for terms that result in compression for positive changes.

The system can be reduced to free body diagrams assuming the spring is in compression.

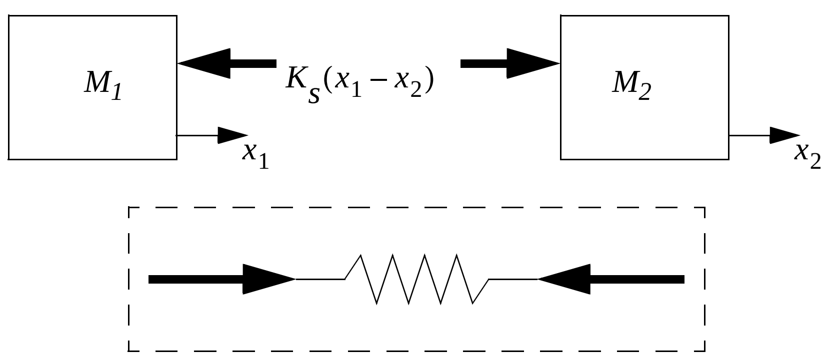


Figure 2.13 Drawing FBDs with interconnecting springs

Sometimes the true length of a spring is important, and the deformation alone is insufficient. In these cases the deformation can be defined as a deformed and undeformed length, as shown in Figure 2.14.

$$\Delta x = l_1 - l_0$$

where,

Δx = *deformation*

l_0 = *the length when undeformed*

l_1 = *the length when deformed*

Figure 2.14 Using the actual spring length

In addition to providing forces, springs may be used as energy storage devices. Figure 2.15 shows the equation for energy stored in a spring.

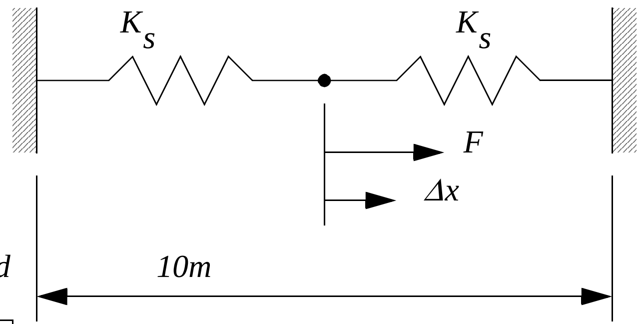
$$E_P = \frac{K(\Delta x)^2}{2}$$

Figure 2.15 Energy stored in a spring

Given,

$$K_s = 10\frac{N}{m}$$

$$\Delta x = 0.1m$$



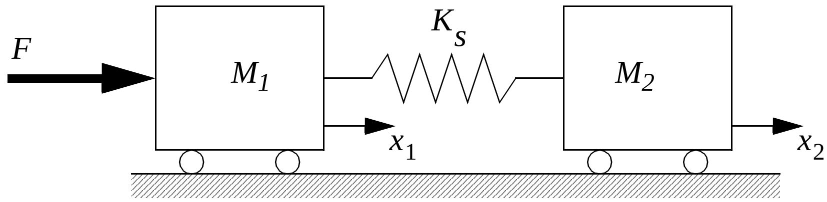
Find F assuming the springs are normally 4m long when unloaded

Aside: it can help to draw a FBD of the pin.

ans. F=2N

Figure 2.16 Drill problem: Deformation of a two spring system

Draw the FBDs and the equations of motion for the masses



ans.

$$\ddot{x}_1(M_1) + x_1(K_s) + x_2(-K_s) = F$$

$$x_1(K_s) + \ddot{x}_2(-M_2) + x_2(-K_s) = F$$

Figure 2.17 Drill problem: Draw the FBDs for the masses

2.2.5 Damping and Drag

A damper is a component that resists motion. The resistive force is relative to the rate of displacement. As mentioned before, springs store energy in a system but dampers dissipate energy. Dampers and springs are often used to compliment each other in designs.

Damping can occur naturally in a system, or can be added by design. The physical damper pictured in Figure 2.18 uses a cylinder that contains a fluid. There is a moving rod and piston that can slide within the cylinder. As the piston moves, fluid is forced through a small orifice. When moved slowly the fluid moves easily, but when moved quickly the pressure required to force the fluid through the orifice rises. This rise in pressure results in a higher force of resistance. In ideal terms any motion would result in an opposing force. In reality there is also a break-away force that needs to be applied before motion begins. Other manufacturing variations could also lead to other small differences between dampers. Normally these cause negligible effects.

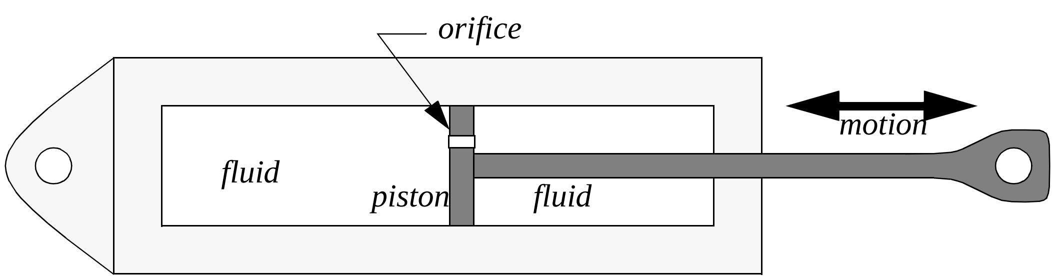
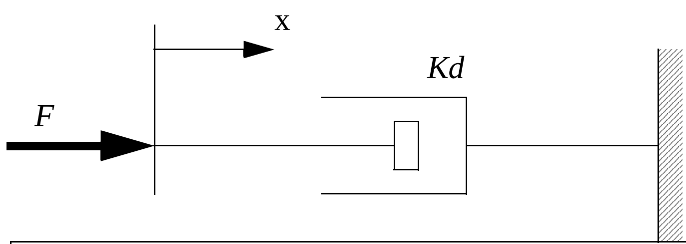


Figure 2.18 A physical damper

The basic equation for an ideal damper in compression is shown in Figure 2.19. In this case the force and displacement are both compressive. The force is calculated by multiplying the damping coefficient by the velocity, or first derivative of position. Aside from the use of the first derivative of position, the analysis of dampers in systems is similar to that of springs.

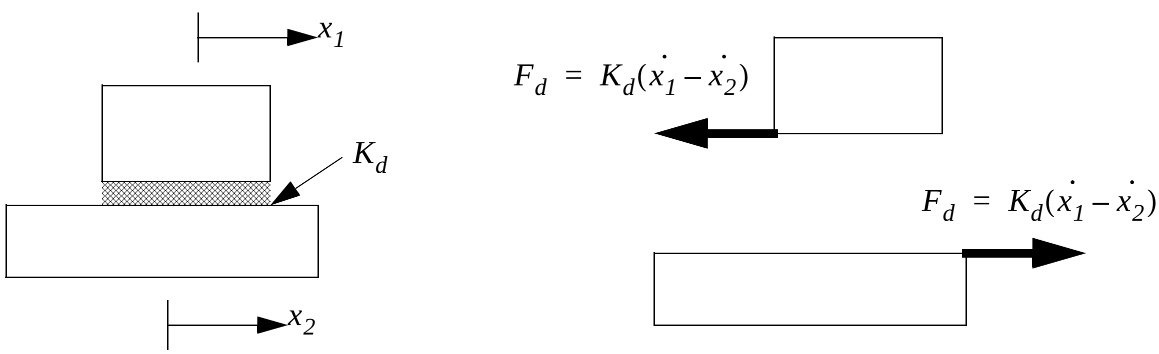


$$F = K_d\left(\frac{d}{dt}\right)x \qquad (15)$$

Aside: The symbol shown is typically used for dampers. It is based on an old damper design called a dashpot. It was constructed using a small piston inside a larger pot filled with oil.

Figure 2.19 An ideal damper

Damping can also occur when there is relative motion between two objects. If the objects are lubricated with a viscous fluid (e.g., oil) then there will be a damping effect. In the example in Figure 2.20 two objects are shown with viscous friction (damping) between them. When the system is broken into free body diagrams the forces are shown to be a function of the relative velocities between the blocks.

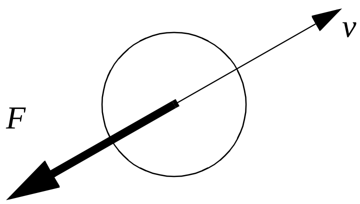


Aside: Fluids, such as oils, have a significant viscosity. When these materials are put in shear they resist the motion. The higher the shear rate, the greater the resistance to flow. Normally these forces are small, except at high velocities.

Figure 2.20 Viscous damping between two bodies with relative motion

A damping force is proportional to the first derivative of position (velocity). Aerodynamic drag is proportional to the velocity squared. The equation for drag is shown in

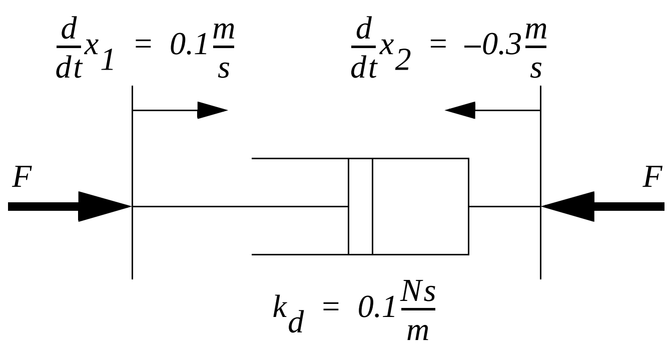
Figure 2.21 in vector and scalar forms. The drag force increases as the square of velocity. Normally, the magnitude of the drag force coefficient 'D' is approximated theoretically and/or measured experimentally. The drag coefficient is a function of material type, surface properties, object size and object geometry.



$$F = -D|v|v$$

Figure 2.21 Aerodynamic drag

The force is acting on the cylinder, resulting in the velocities given below. What is the applied force?



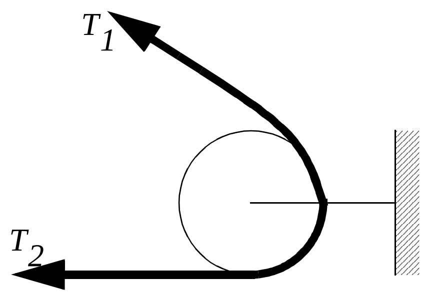
ans. $F = -0.02N$

Figure 2.22 Drill problem: Find the required forces on the damper

2.2.6 Cables And Pulleys

Cables are useful when transmitting tensile forces or displacements. The centerline of the cable becomes the centerline for the force. And, if the force becomes compressive, the cable becomes limp, and will not transmit force. A cable by itself can be represented as a force vector. When used in combination with pulleys, a cable can redirect a force vector or multiply a force.

Typically we assume that a pulley is massless and frictionless (in the rotation chapter we will assume they are not). If this is the case then the tension in the cable on both sides of the pulley are equal, as shown in Figure 2.23.

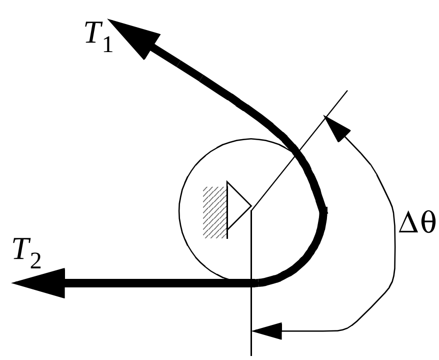


for a massless frictionless pulley

$$T_1 = T_2$$

Figure 2.23 Tension in a cable over a massless frictionless pulley

If we have a pulley that is fixed and cannot rotate, the cable must slide over the surface of the pulley. In this case we can use the coefficient of friction to determine the relative ratio of forces between the sides of the pulley, as shown in Figure 2.24.



$$\frac{T_2}{T_1} = e^{\mu_k(\Delta\theta)}$$

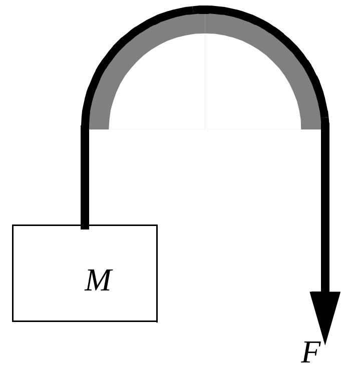
Figure 2.24 Friction of a belt over a fixed drum

Given,

$$\mu_s = 0.35 \qquad \mu_k = 0.2$$

$$M = 1Kg$$

Find F to start the mass moving up and down, and then the force required to maintain a low velocity motion.



ans.

$$F_{up} = 9.81e^{0.2\left(\frac{\pi}{2}\right)} N$$

$$F_{down} = \frac{9.81N}{e^{0.2\left(\frac{\pi}{2}\right)}}$$

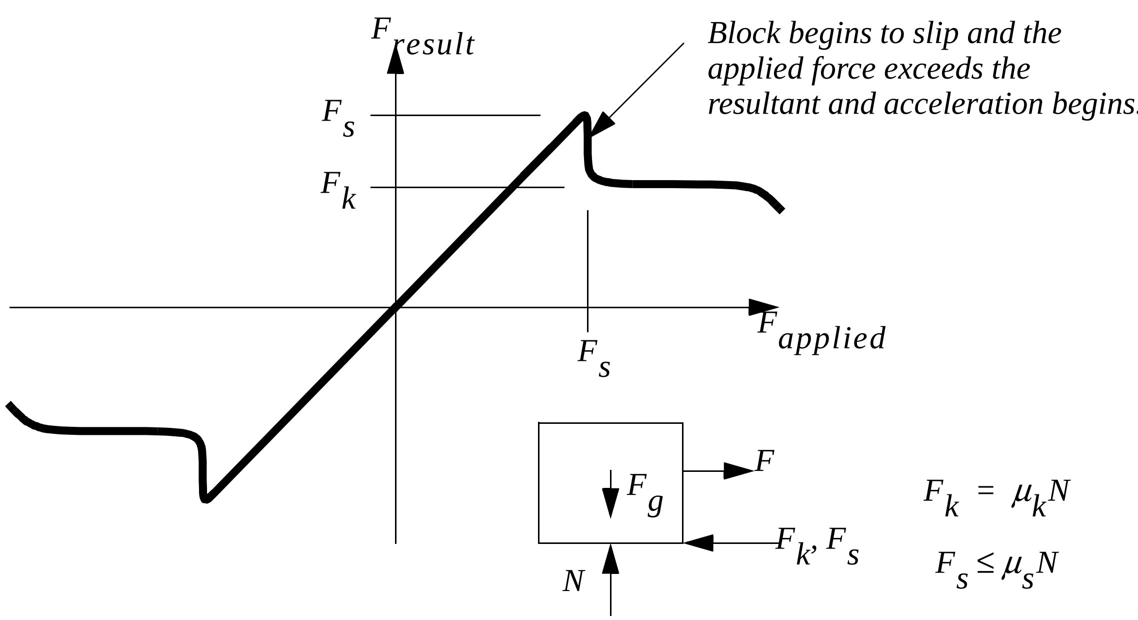
Figure 2.25 Drill problem: Friction forces for belts on drums

Although the discussion in this section has focused on cables and pulleys, the theory also applies to belts over drums.

2.2.7 Friction

Viscous friction was discussed before, where a lubricant would provide a damping effect between two moving objects. In cases where there is no lubricant, and the touching surfaces are dry, dry coulomb friction may result. In this case the surfaces will stick in place until a maximum force is overcome. After that the object will begin to slide and a constant friction force will result.

Figure 2.26 shows the classic model for (dry Coulomb) friction. The force on the horizontal axis is the force applied to the friction surfaces while the vertical axis is the resulting friction force. Beneath the slip force the object will stay in place. When the slip force is exceeded the object will begin to move, and the resulting kinetic friction force will be relatively constant. (Note: If the object begins to travel much faster then the kinetic friction force will decrease.) It is common to forget that friction forces are bidirectional, but it always opposes the applied force or motion. The friction force is a function of the coefficient of friction and the normal force across the contact surfaces. The coefficient of friction is a function of the materials, surface texture and surface shape.

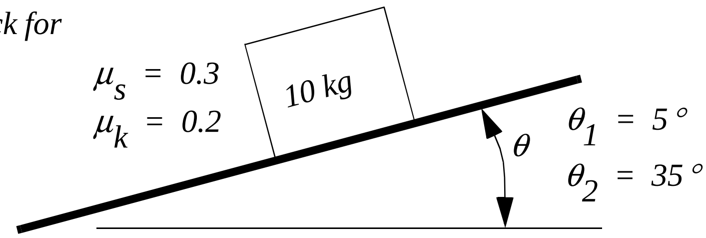


Note: When solving problems with friction remember that the friction force will always equal the applied force (not the maximum force) until slip occurs. After that the friction is approximately constant. In addition, the friction forces direction opposes applied forces, and motion.

Figure 2.26 Dry friction

Many systems use kinetic friction to dissipate energy from a system as heat, sound and vibration.

Find the acceleration of the block for both angles indicated.



ans.

Figure 2.27 Drill problem: find the accelerations

2.2.8 Contact Points And Joints

A system is built by connecting components together. These connections can be rigid or moving. In solid connections all forces and moments are transmitted and the two pieces act as a single rigid body. In moving connections there is at least one degree of freedom. If we limit this to translation only, there are up to three degrees of freedom, x, y and z. In any direction there is a degree of freedom, a force or moment cannot be transmitted.

When constructing FBDs for a system we must break all of the components into individual rigid bodies. Where the mechanism has been broken the contact forces must be added to both of the separated pieces. Consider the example in Figure 2.28. At joint A the forces are written as two components in the x and y directions. For joint B the force components with equal magnitudes but opposite directions are added to both FBDs.

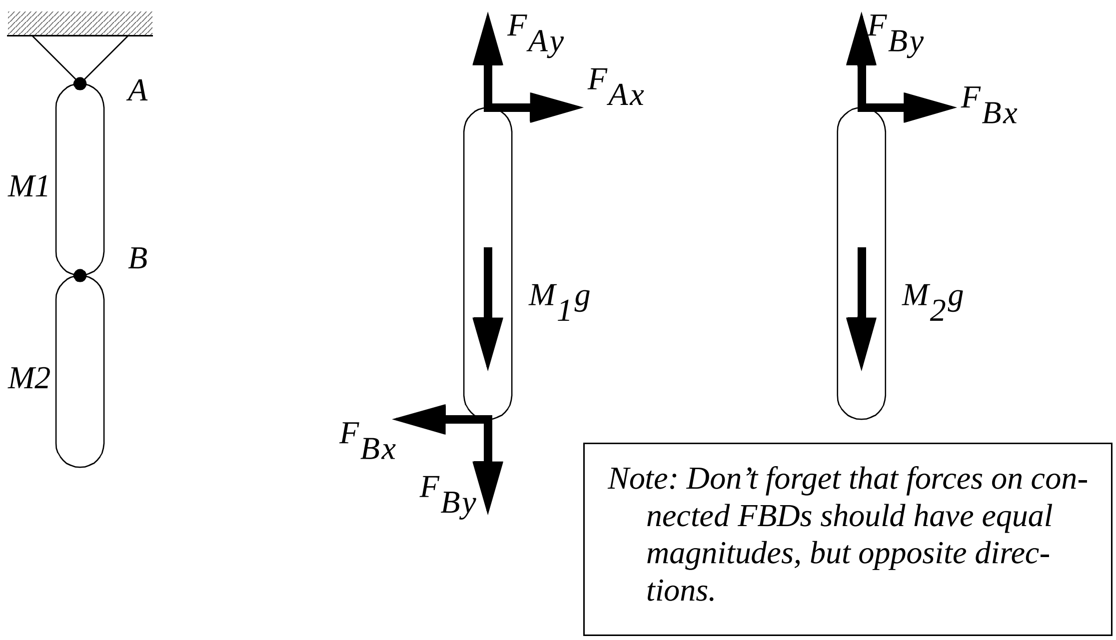


Figure 2.28 FBDs for systems with connected members

2.3 SYSTEM EXAMPLES

An orderly approach to system analysis can simplify the process of analyzing large systems. The list of steps below is based on general observations of good problem solving techniques.

1. Assign letters/numbers to designate force components (if not already done) - this will allow you to refer to components in your calculations.
2. Define positions and directions for any moving masses. This should include the selection of reference points.
3. Draw free body diagrams for each component, and add forces (inertia is optional).
4. Write equations for each component by summing forces.
5. (next chapter) Combine the equations by eliminating unwanted variables.
6. (next chapter) Develop a final equation that relates input (forcing functions) to outputs (results).

Note: When deriving differential equations, the final value can be checked for errors using unit analysis. This method involves replacing variables with their unit equivalents. All the units should match.

coefficient	units
F	$N = \frac{Kgm}{s^2}$
K_s	$\frac{N}{m}$
K_d	$\frac{Ns}{m}$
M	Kg

e.g., $\ddot{x}_2(M_2) + \dot{x}_2(B) + x_2(K_{s2}) + x_1(-K_{s2}) = F$

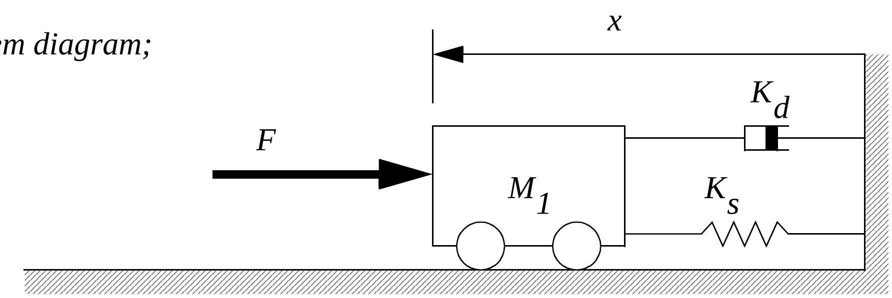
$$\therefore \frac{m}{s^2}(Kg) + \frac{m}{s}\left(\frac{Ns}{m}\right) + m\left(\frac{N}{m}\right) + m\left(-\frac{N}{m}\right) = N$$

$$\therefore N + (N) + (N) + (-N) = N$$

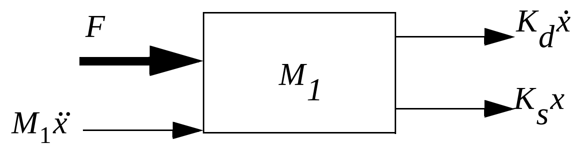
The units match, so there are no obvious problems.

Consider the cart in Figure 2.29. On the left is a force, it is opposed by a spring and damper on the right. The basic problem definition already contains all of the needed definitions, so no others are required. The FBD for the mass shows the applied force and the reaction forces from the spring and damper. When the forces are summed the inertia is on the right side of the equation in Newton’s form. This equation is then rearranged to a second-order non-homogeneous differential equation.

Given the system diagram;



The FBD for the cart is



The forces for the cart are in a single direction and can be summed as,

$$\overset{+}{\leftarrow} \quad \sum F_x = -F - K_d\dot{x} - K_s x = M_1\ddot{x}$$

This equation can be rearranged to a second-order non-homogeneous diff. eqn.

$$\ddot{x} + \frac{K_d}{M_1}\dot{x} + \frac{K_s}{M_1}x = \frac{F}{M_1}$$

Aside: later on we will solve the differential equations, or use other methods to determine how the system will behave. It is useful to have all of the 'output' variables for the system on the left hand side, and everything else on the other.

Figure 2.29 A simple translational system example

Develop the equation relating the input force to the motion (in terms of x) of the lefthand cart for the problem below.

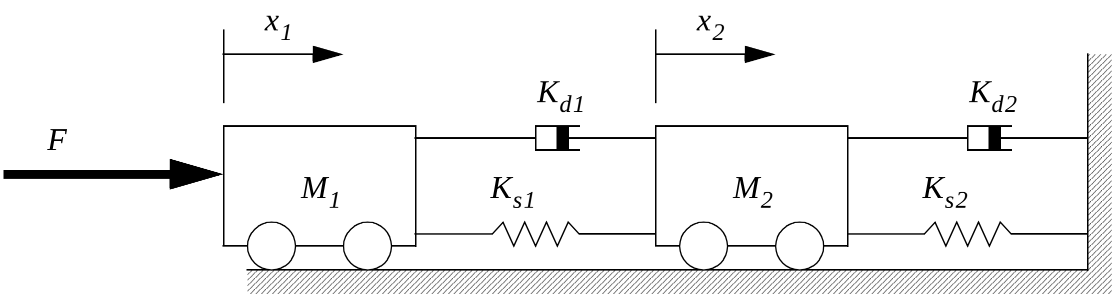


Figure 2.30 Drill problem: Find the differential equations

A simplified model of an elevator (M1) and a passenger (M2) are shown in Figure 2.31. In this example many of the required variables need to be defined. These are added to the FBDs. Care is also taken to ensure that all forces between bodies are equal in magnitude, but opposite in direction. The wall forces are ignored because they are statically indeterminate, and x-axis force components are irrelevant to the forces in the y-axis.

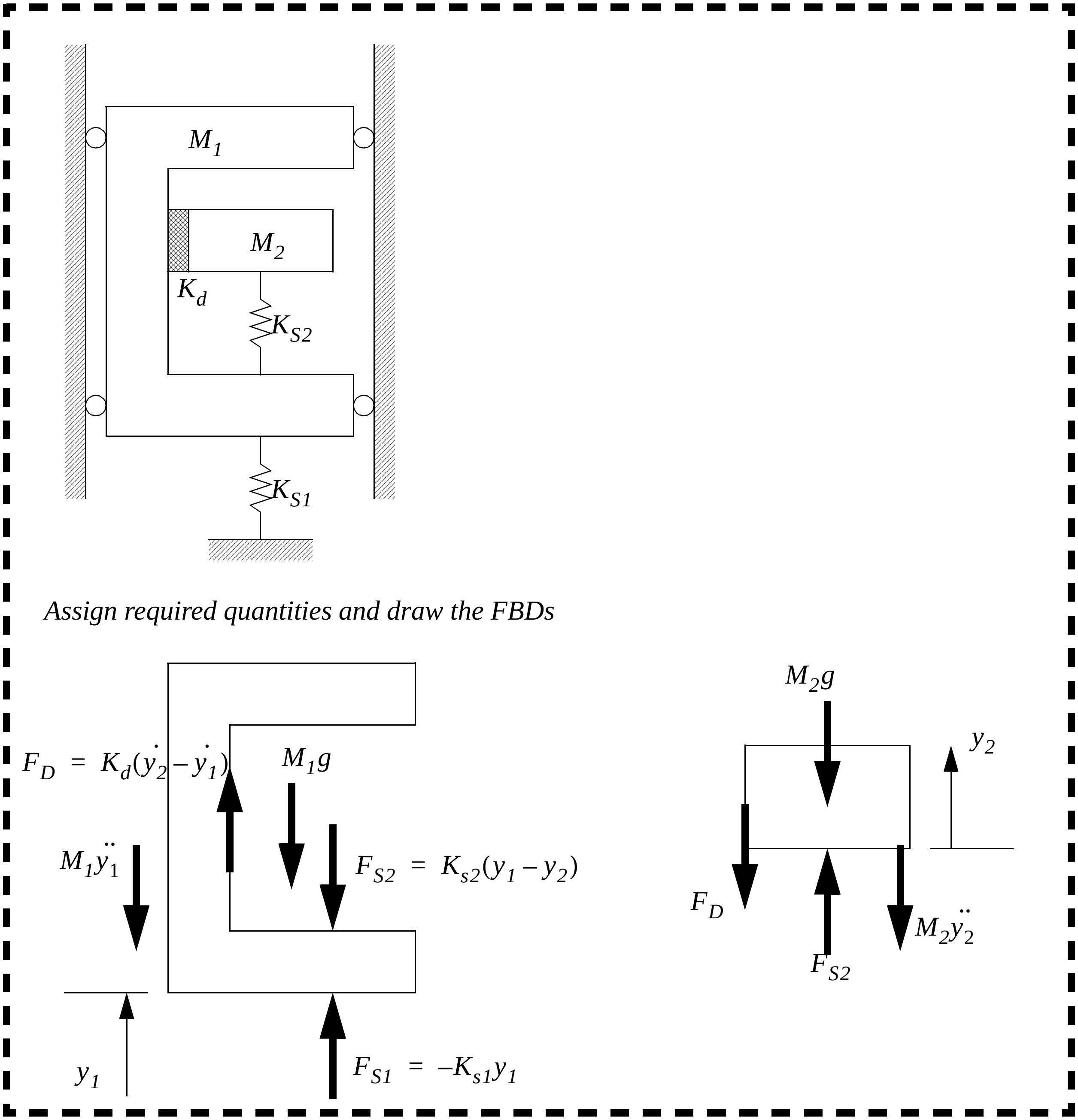


Figure 2.31 A multi-body translating system (an elevator with a passenger)

The forces on the FBDs are summed and the equations are expanded in Figure 2.32.

Now, sum the forces in vector form, and substitute relationships,

$$\sum F_{M_1} = F_{S1} + F_{S2} + M_1 g + F_D = M_1 a_1$$

$$\sum F_{M_2} = -F_{S2} + M_2 g - F_D = M_2 a_2$$

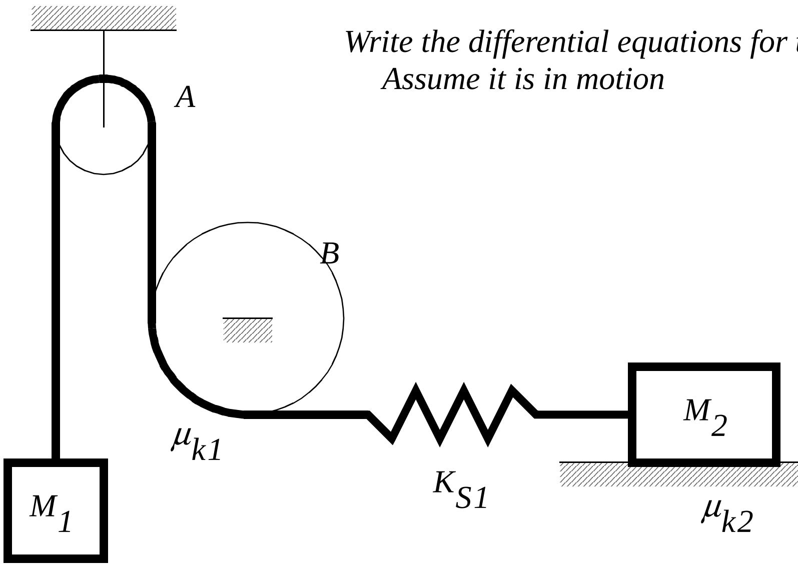
At this point the equations are expanded

$$K_{S1}(-y_1) + K_{S2}(y_2 - y_1) + M_1(-9.81) + K_d \frac{d}{dt}(y_2 - y_1) = M_1 a_y$$

$$-K_{S2}(y_2 - y_1) + M_2(-9.81) - K_d \frac{d}{dt}(y_2 - y_1) = M_2 a_y$$

Figure 2.32 Equations for the elevator

Write the differential equations for the system below.
Assume it is in motion



ans.

Figure 2.33 Drill problem: A more complex translational systems

Consider the springs shown in Figure 2.34. When two springs are combined in this manner they can be replaced with a single equivalent spring. In the parallel spring combination the overall stiffness of the spring would increase. In the series spring combination the overall stiffness would decrease.

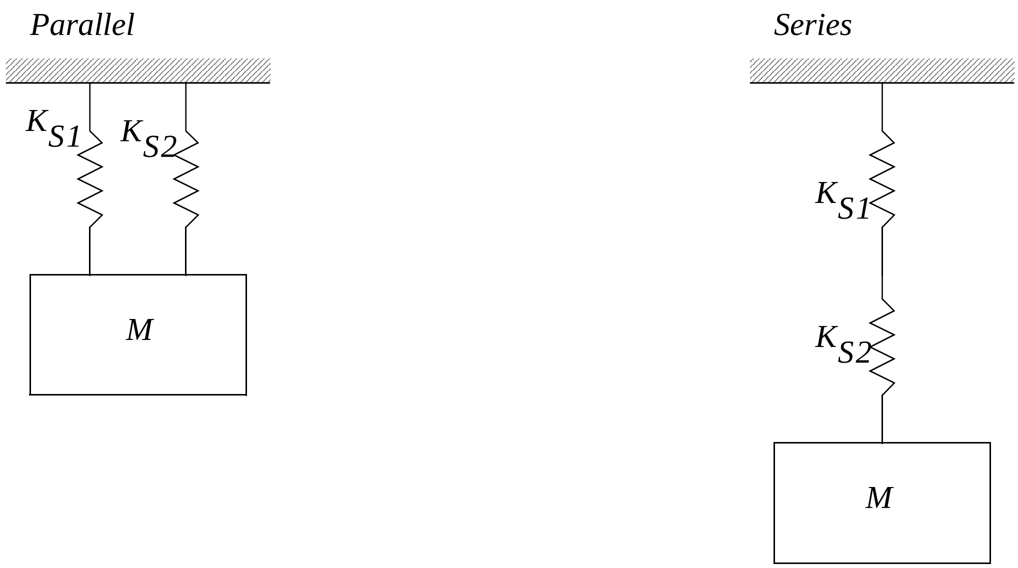
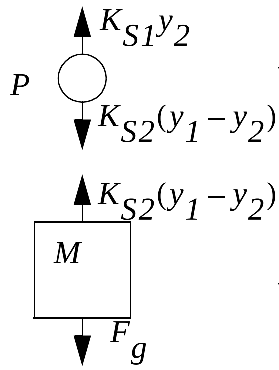


Figure 2.34 Springs in parallel and series (kinematically)

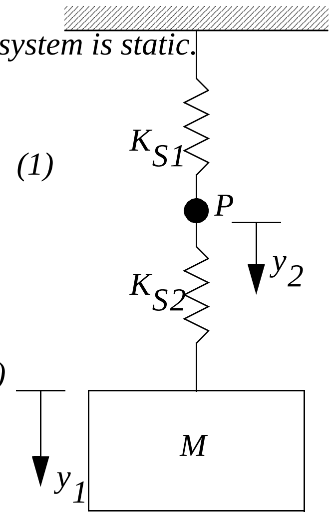
Figure 2.35 shows the calculations required to find a spring coefficient equivalent to the two springs in series. The first step is to draw a FBD for the mass at the bottom, and for a point between the two springs, P. The forces for both of these are then summed. The next process is to combine the two equations to eliminate the height variable created for point P. After this, the equation is rearranged into Hooke's law, and the equivalent spring coefficient is found.

First, draw FBDs for P and M and sum the forces assuming the system is static.



$$+\uparrow \sum F_y = K_{S1}y_2 - K_{S2}(y_1 - y_2) = 0 \quad (1)$$

$$+\uparrow \sum F_y = K_{S2}(y_1 - y_2) - F_g = 0 \quad (2)$$



Next, rearrange the equations to eliminate y2 and simplify.

(1) becomes $$K_{S2}(y_1 - y_2) - F_g = 0$$

$$y_1 - y_2 = \frac{F_g}{K_{S2}}$$

$$y_2 = y_1 - \frac{F_g}{K_{S2}} \quad (3)$$

(2) becomes $$K_{S1}y_2 - K_{S2}(y_1 - y_2) = 0$$

$$y_2(K_{S1} + K_{S2}) = y_1 K_{S2} \quad (4)$$

sub (3) into (4) $$\left(y_1 - \frac{F_g}{K_{S2}}\right)(K_{S1} + K_{S2}) = y_1 K_{S2}$$

$$y_1 - \frac{F_g}{K_{S2}} = y_1 \frac{K_{S2}}{K_{S1} + K_{S2}}$$

$$F_g = y_1\left(1 - \frac{K_{S2}}{K_{S1} + K_{S2}}\right)K_{S2}$$

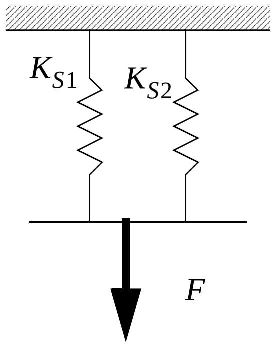
$$F_g = y_1\left(\frac{K_{S1} + K_{S2} - K_{S2}}{K_{S1} + K_{S2}}\right)K_{S2}$$

$$F_g = y_1\left(\frac{K_{S1}K_{S2}}{K_{S1} + K_{S2}}\right)$$

Finally, consider the basic spring equation to find the equivalent spring coefficient.

$$K_{equiv} = \frac{K_{S1}K_{S2}}{K_{S1} + K_{S2}}$$

Figure 2.35 Calculation of an equivalent spring coefficient for springs in series



ans.

Figure 2.36 Drill problem: Find an equivalent spring for the springs in parallel

Consider the drill problem. When an object has no mass, the force applied to one side of the spring will also be applied to the other. The only factor that changes is displacement.

Show that a force applied to one side of a massless spring is the reaction force at the other side.

ans.

Figure 2.37 Drill problem: Prove that the force on both sides is equal

2.4 OTHER TOPICS

Designing a system in terms of energy content can allow insights not easily obtained by the methods already discussed. Consider the equations in Figure 2.38. These equations show that the total energy in the system is the sum of kinetic and potential energy. Kinetic energy is half the product of mass times velocity squared. Potential energy in translating systems is a force magnitude multiplied by a distance (that force was applied over). In addition, the power, or energy transfer rate is the force applied multiplied by the velocity.

$$E = E_P + E_K \quad (7)$$

$$E_K = \frac{Mv^2}{2} \quad (8)$$

$$E_P = Fd = Mgd \quad (9)$$

$$P = Fv = \frac{d}{dt}E \quad (10)$$

Figure 2.38 Energy and power equations for translating masses

2.5 SUMMARY

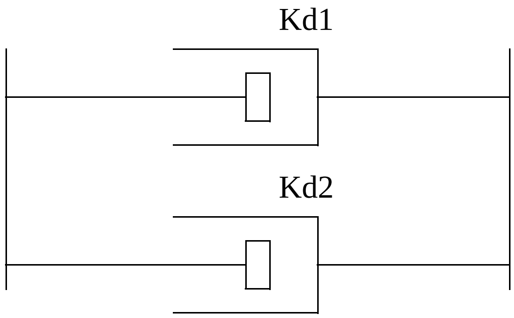
- FBDs are useful for reducing complex systems to simpler parts.
- Equations for translation and rotation can be written for FBDs.
- The equations can be integrated for dynamic cases, or solved algebraically for static cases.

2.6 PRACTICE PROBLEMS

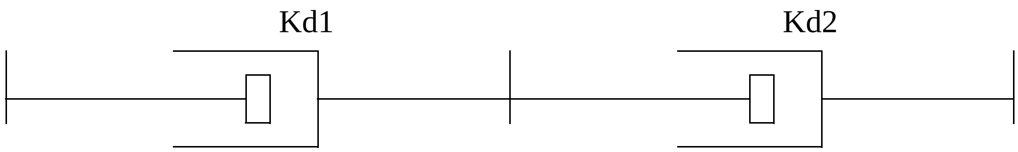
1. If a spring has a deflection of 6 cm when exposed to a static load of 200N, what is the spring constant?

2. Derive the effective damping coefficients for the pairs below from basic principles,

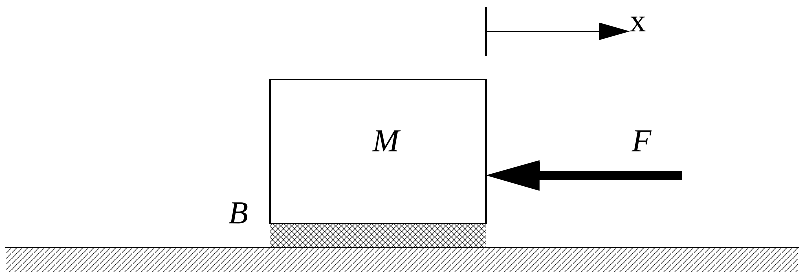
a)



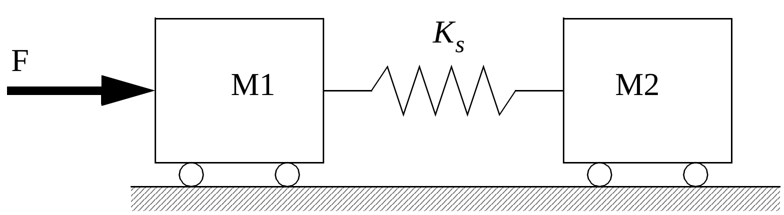
b)



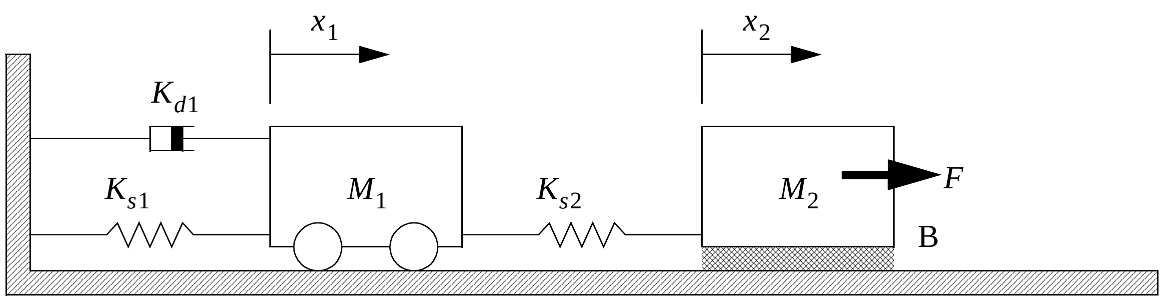
3. Write a differential equation for the mass pictured below.



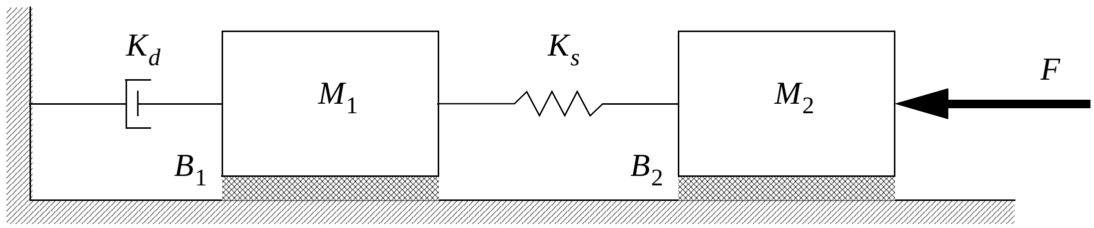
4. Write the differential equations for the translating system below.



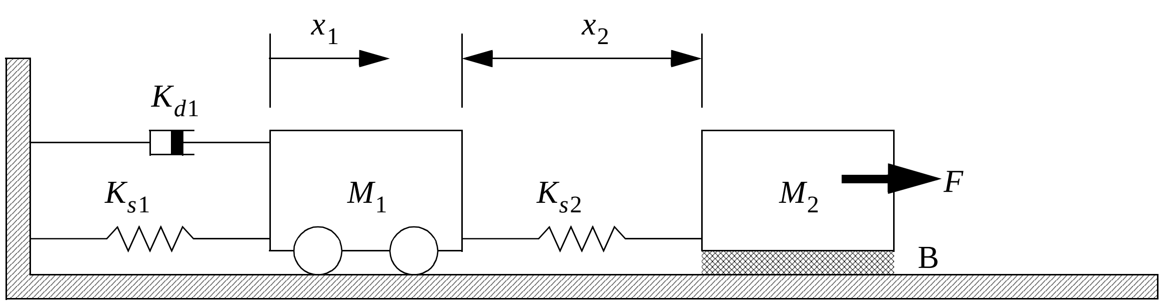
5. Write the differential equations for the system below.



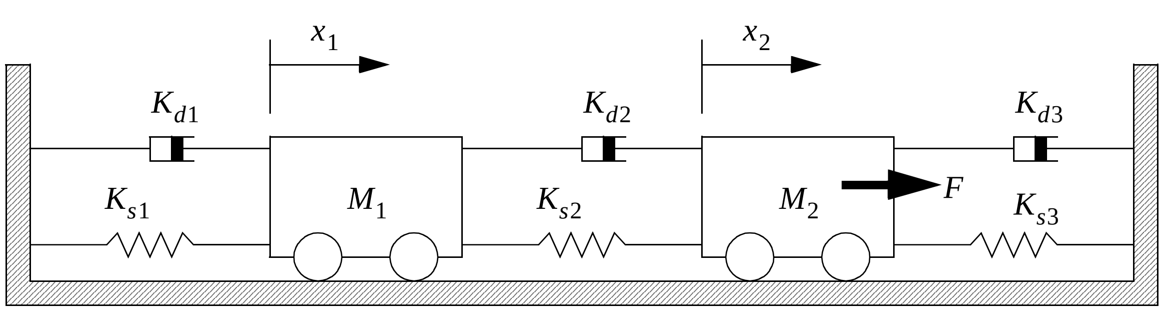
6. Write the differential equations for the system given below.



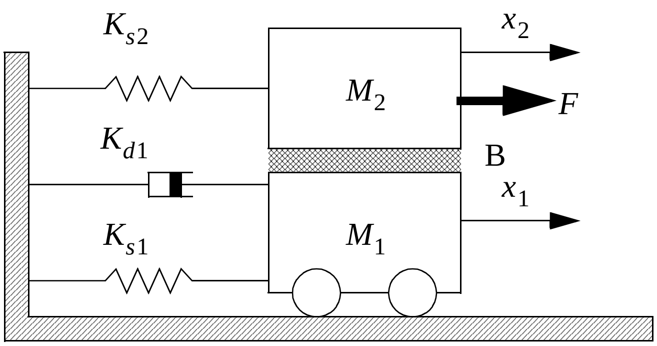
7. Write the differential equations for the system below.



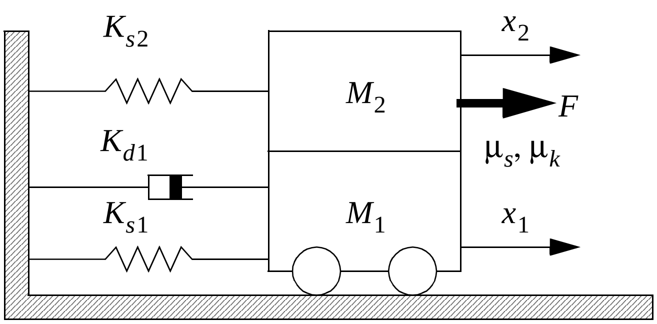
8. Write the differential equations for the system below.



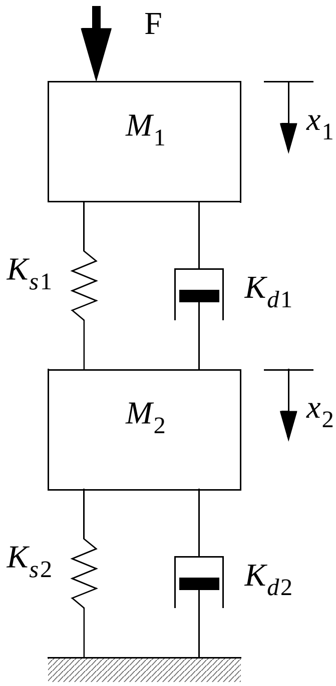
9. Write the differential equations for the system below.



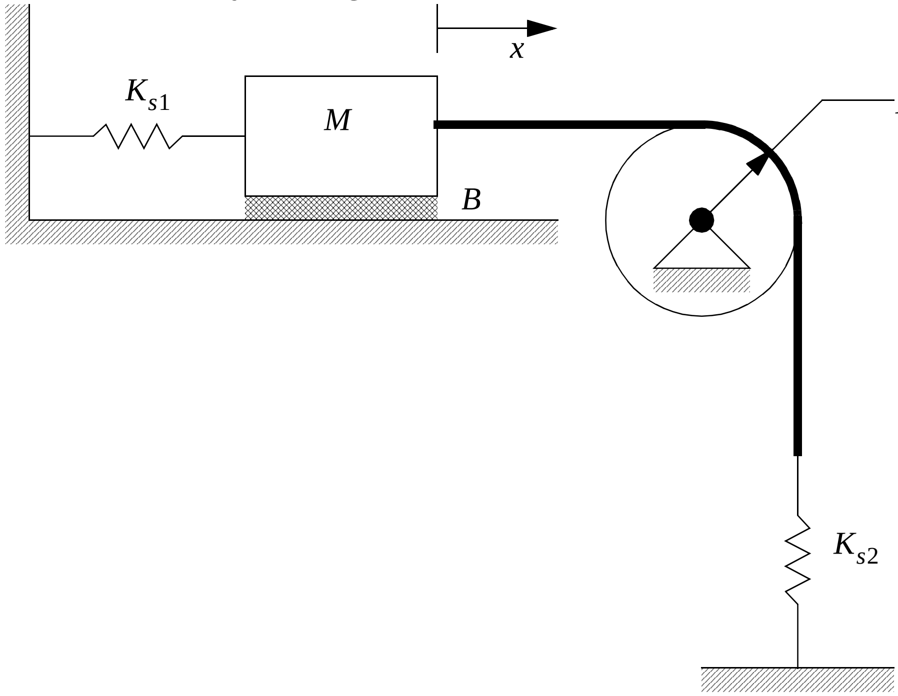
10. Write the differential equations for the system below.



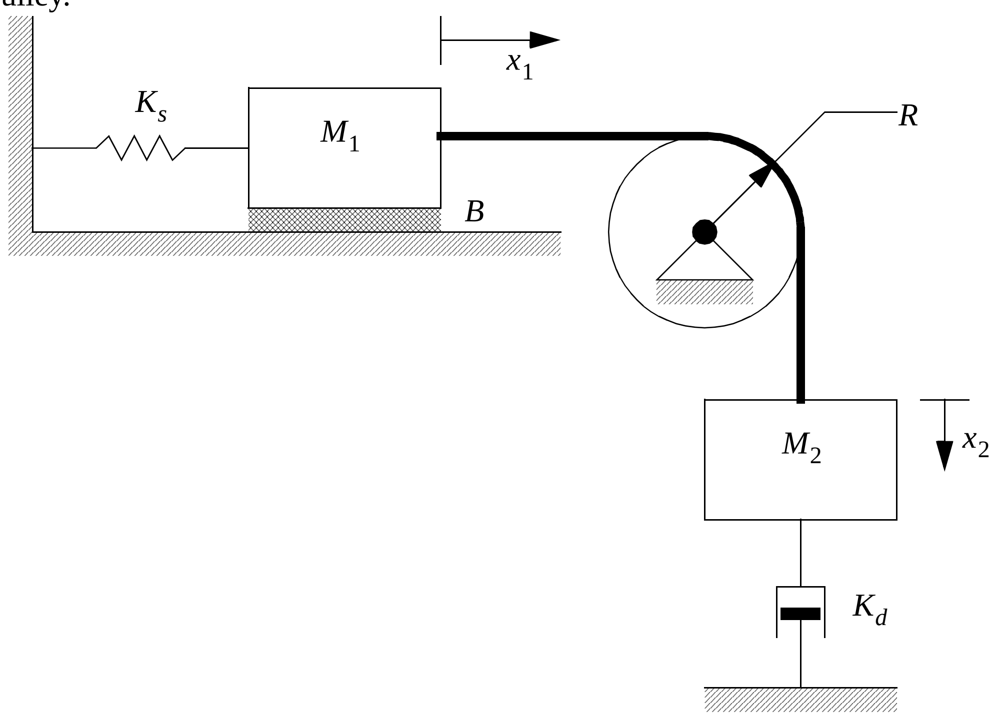
11. Write the differential equations for the system below.



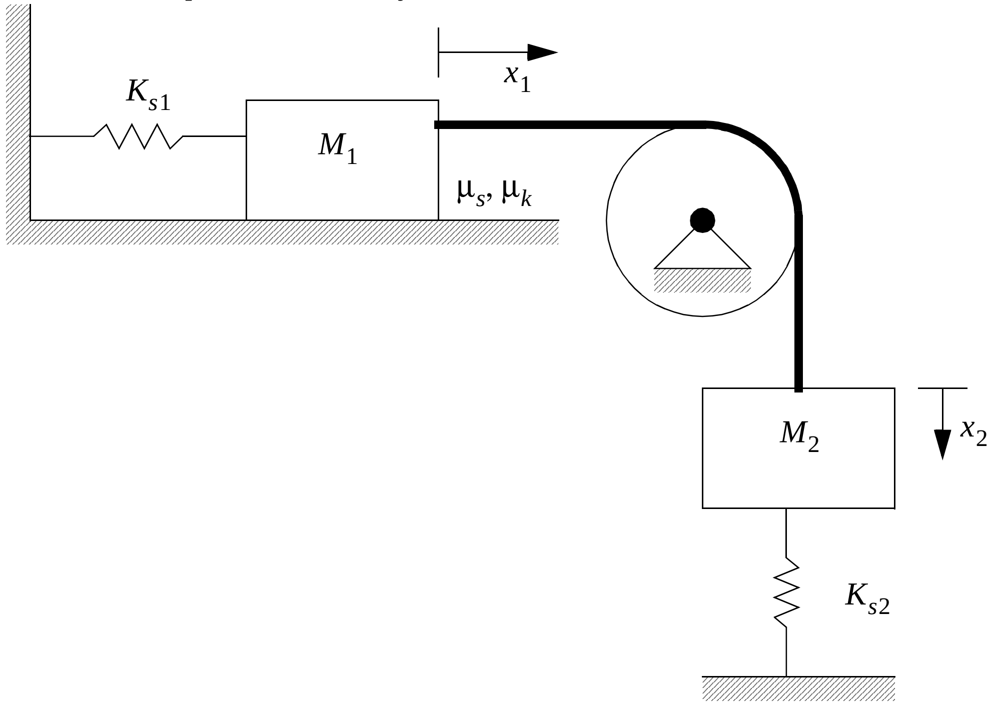
12. Write the differential equations for the system below. Assume that the pulley is massless and frictionless and that the system begins undeflected.



13. Write the differential equations for the system below. In this system the upper mass, M1, is between a spring and a cable and there is viscous damping between the mass and the floor. The suspended mass, M2, is between the cable and a damper. The cable runs over a massless, frictionless pulley.



14. Write the differential equations for the system below.



2.7 PRACTICE PROBLEM SOLUTIONS

1. Ks = 33.3 N/cm = 3333 N/m

2.

a) $K_{eq} = K_{d1} + K_{d2}$

b) $K_{eq} = \frac{K_{d1}K_{d2}}{K_{d1} + K_{d2}}$

3.

$$\ddot{x} + \dot{x}\left(\frac{B}{M}\right) = -\frac{F}{M}$$

4.

$$\ddot{x}_1 + x_1\left(\frac{K_s}{M_1}\right) + x_2\left(\frac{-K_s}{M_1}\right) = \frac{F}{M_1}$$

$$\ddot{x}_2 + x_2\left(\frac{K_s}{M_2}\right) + x_1\left(\frac{-K_s}{M_2}\right) = 0$$

5.

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_{d1}}{M_1}\right) + x_1\left(\frac{K_{s1} + K_{s2}}{M_1}\right) + x_2\left(\frac{-K_{s2}}{M_1}\right) = 0$$

$$\ddot{x}_2 + \dot{x}_2\left(\frac{B}{M_2}\right) + x_2\left(\frac{K_{s2}}{M_2}\right) + x_1\left(\frac{-K_{s2}}{M_2}\right) = \frac{F}{M_2}$$

6.

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_d + B_1}{M_1}\right) + x_1\left(\frac{K_s}{M_1}\right) + x_2\left(\frac{-K_s}{M_1}\right) = 0$$

$$\ddot{x}_2 + \dot{x}_2\left(\frac{B_2}{M_2}\right) + x_2\left(\frac{K_s}{M_2}\right) + x_1\left(\frac{-K_s}{M_2}\right) = \frac{-F}{M_2}$$

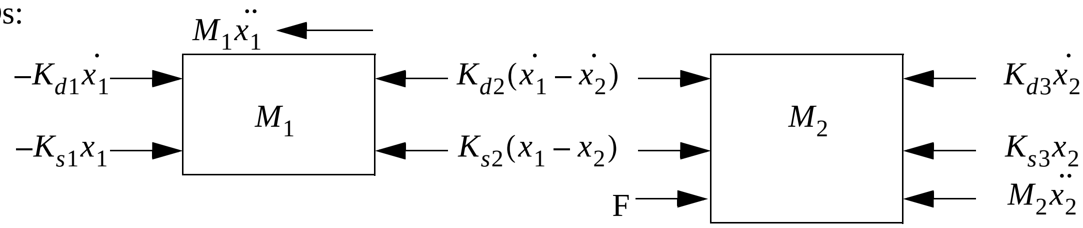
7.

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_{d1}}{M_1}\right) + x_1\left(\frac{K_{s1}}{M_1}\right) + x_2\left(\frac{-K_{s2}}{M_1}\right) = 0$$

$$\ddot{x}_1 + \dot{x}_1\left(\frac{B}{M_2}\right) + \ddot{x}_2 + \dot{x}_2\left(\frac{B}{M_2}\right) + x_2\left(\frac{K_s}{M_2}\right) = \frac{F}{M_2}$$

8.

FBDs:



For M1: $\xrightarrow{+}$

$$\sum F = -K_{d1}\dot{x}_1 - K_{s1}x_1 - K_{d2}(\dot{x}_1 - \dot{x}_2) - K_{s2}(x_1 - x_2) = M_1\ddot{x}_1$$

$$\ddot{x}_1(M_1) + \dot{x}_1(K_{d1} + K_{d2}) + x_1(K_{s1} + K_{s2}) + \dot{x}_2(-K_{d2}) + x_2(-K_{s2}) = 0$$

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_{d1} + K_{d2}}{M_1}\right) + x_1\left(\frac{K_{s1} + K_{s2}}{M_1}\right) + \dot{x}_2\left(\frac{-K_{d2}}{M_1}\right) + x_2\left(\frac{-K_{s2}}{M_1}\right) = 0$$

For M2: $\xrightarrow{+}$

$$\sum F = K_{d2}(\dot{x}_1 - \dot{x}_2) + K_{s2}(x_1 - x_2) + F - K_{d3}\dot{x}_2 - K_{s3}x_2 = M_2\dot{x}_2$$

$$\ddot{x}_2(M_2) + \dot{x}_2(K_{d2} + K_{d3}) + x_2(K_{s2} + K_{s3}) + \dot{x}_1(-K_{d2}) + x_1(-K_{s2}) = F$$

$$\ddot{x}_2 + \dot{x}_2\left(\frac{K_{d2} + K_{d3}}{M_2}\right) + x_2\left(\frac{K_{s2} + K_{s3}}{M_2}\right) + \dot{x}_1\left(\frac{-K_{d2}}{M_2}\right) + x_1\left(\frac{-K_{s2}}{M_2}\right) = F$$

9.

$$\ddot{x}_1(M_1) + \dot{x}_1(K_{d1} + B) + x_1(K_{s1}) + \dot{x}_2(-B) = 0$$

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_{d1} + B}{M_1}\right) + x_1\left(\frac{K_{s1}}{M_1}\right) + \dot{x}_2\left(\frac{-B}{M_1}\right) = 0$$

$$\ddot{x}_2(M_2) + \dot{x}_2(B) + x_2(K_{s2}) + \dot{x}_1(-B) = F$$

$$\ddot{x}_2 + \dot{x}_2\left(\frac{B}{M_2}\right) + x_2\left(\frac{K_{s2}}{M_2}\right) + \dot{x}_1\left(\frac{-B}{M_2}\right) = \frac{F}{M_2}$$

10.

$$\ddot{x}_1(M_1) + \dot{x}_1(K_{d1}) + x_1(K_{s1}) = F_F$$

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_{d1}}{M_1}\right) + x_1\left(\frac{K_{s1}}{M_1}\right) = \frac{F_F}{M_1}$$

$$\ddot{x}_2(M_2) + x_2(K_{s2}) = F - F_F$$

$$\ddot{x}_2 + x_2\left(\frac{K_{s2}}{M_2}\right) = \frac{F - F_F}{M_2}$$

where, $|F_F| \le \mu_s M_2 g$ if $\dot{x}_1 = \dot{x}_2$

$$F_F = \mu_k M_2 g\left(\frac{\dot{x}_1 - \dot{x}_2}{|\dot{x}_1 - \dot{x}_2|}\right) \quad \text{if} \quad \dot{x}_1 \ne \dot{x}_2$$

11.
(assuming no gravity

$$\ddot{x}_1(M_1) + \dot{x}_1(K_{d1}) + x_1(K_{s1}) + \dot{x}_2(-K_{d1}) + x_2(-K_{s1}) = F$$

$$\ddot{x}_1 + \dot{x}_1\left(\frac{K_{d1}}{M_1}\right) + x_1\left(\frac{K_{s1}}{M_1}\right) + \dot{x}_2\left(\frac{-K_{d1}}{M_1}\right) + x_2\left(\frac{-K_{s1}}{M_1}\right) = \frac{F}{M_1}$$

$$\ddot{x}_2(M_2) + \dot{x}_2(K_{d1} + K_{d2}) + x_2(K_{s1} + K_{s2}) + \dot{x}_1(-K_{d1}) + x_1(-K_{s1}) = 0$$

$$\ddot{x}_2 + \dot{x}_2\left(\frac{K_{d1} + K_{d2}}{M_2}\right) + x_2\left(\frac{K_{s1} + K_{s2}}{M_2}\right) + \dot{x}_1\left(\frac{-K_{d1}}{M_2}\right) + x_1\left(\frac{-K_{s1}}{M_2}\right) = 0$$

12.

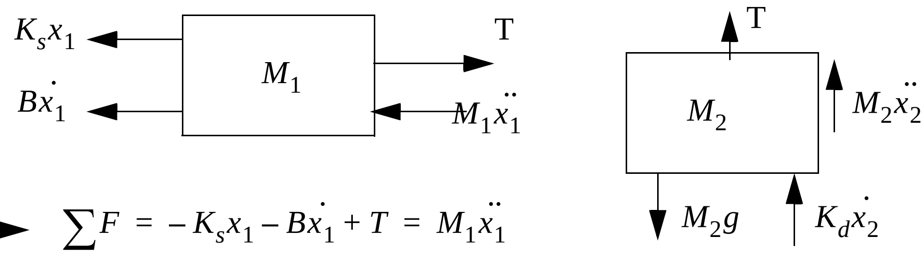
$$if(x \ge 0) \qquad T = 0$$

$$if(x < 0) \qquad T = -K_{s2}x$$

$$\ddot{x} + \dot{x}\left(\frac{B}{M_1}\right) + x\left(\frac{K_{s1}}{M_1}\right) = \frac{T}{M_1}$$

13.

FBDs:



For M1: $\xrightarrow{+}$

$$\sum F = -K_s x_1 - B\dot{x}_1 + T = M_1\ddot{x}_1$$

$$\ddot{x}_1(M_1) + \dot{x}_1(B) + x_1(K_s) = T$$

$$\ddot{x}_1 + \dot{x}_1\left(\frac{B}{M_1}\right) + x_1\left(\frac{K_s}{M_1}\right) = \frac{T}{M_1}$$

For M2: $+\uparrow$

$$\sum F = T + K_d\dot{x}_2 - M_2 g = -M_2\ddot{x}_2$$

$$\ddot{x}_2(-M_2) + \dot{x}_2(-K_d) = T - M_2 g$$

$$\ddot{x}_2 + \dot{x}_2\left(\frac{K_d}{M_2}\right) = \frac{T - M_2 g}{-M_2}$$

For T: if T <= 0 then T = 0N
if T > 0 x1 = x2

14.

$if((x_2 - x_1) < 0) \qquad T = 0$

$if(\dot{x}_1 = 0) \qquad |F_F| \le M_1 g \mu_s$

$if(\dot{x}_1 \ne 0) \qquad F_F = M_1 g \mu_k \frac{\dot{x}_1}{|\dot{x}_1|}$

$$\ddot{x}_1 + x_1\left(\frac{K_{s1}}{M_1}\right) = \frac{T - F_F}{M_1}$$

$$\ddot{x}_2 + x_2\left(\frac{K_{s2}}{M_2}\right) = \frac{-T}{M_2} - g$$

2.8 ASSIGNMENT PROBLEMS