6. INPUT-OUTPUT EQUATIONS

Topics:
- The differential operator, input-output equations
- Design case - vibration isolation

Objectives:
- To be able to develop input-output equations for mechanical systems.

6.1 INTRODUCTION

To solve a set of differential equations we have two choices, solve them numerically or symbolically. For a symbolic solution the system of differential equations must be manipulated into a single differential equation. In this chapter we will look at methods for manipulating differential equations into useful forms.

6.2 THE DIFFERENTIAL OPERATOR

The differential operator 'd/dt' can be written in a number of forms. In this book there have been two forms used thus far, d/dt x and x-dot. For convenience we will add a third, 'D'. The basic definition of this operator, and related operations are shown in Figure 6.1. In basic terms the operator can be manipulated as if it is a normal variable. Multiplying by 'D' results in a derivative, dividing by 'D' results in an integral. The first-order axiom can be used to help solve a first-order differential equation.

basic definition	$\frac{d}{dt}x = Dx$ $\qquad \frac{d^n}{dt^n} = D^n$ $\qquad \frac{1}{D}x = \int x\,dt$
algebraic manipulation	$Dx + Dy = D(x + y)$
	$Dx + Dy = Dy + Dx$
	$Dx + (Dy + Dz) = (Dx + Dy) + Dz$
simplification	$\frac{D^n x}{D^m} = D^{n-m}x$
	$\frac{x(D + a)}{(D + a)} = x$
first-order axiom	$\frac{x(t)}{(D + a)} = e^{-at}\left(\int x(t)e^{at}dt + C\right)$

Figure 6.1 General properties of the differential operator

Note:

$$\dot{x} + Ax = y(t)$$

$$xD + Ax = y(t)$$

$$xD + Ax = y(t)$$
$$x(D + A) = y(t)$$
$$x = \frac{y(t)}{D + A}$$

$$e^{At}xD + e^{At}Ax = e^{At}y(t)$$

$$e^{At}xD = e^{At}y(t)$$

$$\frac{d}{dt}(e^{at}x) = e^{at}\frac{d}{dt}x + ae^{at}x$$
$$De^{at}x = e^{at}Dx + ae^{at}x$$

$$e^{At}xD = e^{At}y(t)$$

$$e^{At}x = \int e^{At}y(t)dt + C$$

$$x = e^{-At}\left(\int e^{At}y(t)dt + C\right)$$

$$\frac{y(t)}{D + A} = e^{-At}\left(\int e^{At}y(t)dt + C\right)$$

Figure 6.2 Proof of the first-order axiom

Figure 6.3 contains an example of the manipulation of a differential equation using the 'D' operator. The solution begins by replacing the 'd/dt' terms with the 'D' operator. After this the equation is rearranged to simplify the expression. Notice that the manipulation follows the normal rules of algebra.

$$\left(\frac{d}{dt}\right)^2 x + \frac{d}{dt}x + 5x = 5t$$

$$D^2x + Dx + 5x = 5t$$

$$x(D^2 + D + 5) = 5t$$

$$x = \frac{5t}{D^2 + D + 5}$$

$$x = t\left(\frac{5}{D^2 + D + 5}\right)$$

Figure 6.3 An example of simplification with the differential operator

An example of the solution of a first-order differential equation is given in Figure 6.4. This begins by replacing the differential operator and rearranging the equation. The first-order axiom is then used to obtain the solution. The initial conditions are then used to calculate the coefficient values.

Given, $\frac{d}{dt}x + 5x = 3t \qquad x(0) = 10$

$$Dx + 5x = 3t$$

$$x = \frac{3t}{D+5}$$

$$x = e^{-5t}\left(\int e^{5t} 3t\,dt + C\right)$$

$$x = e^{-5t}\left(3\left(\frac{te^{5t} - e^{5t}}{5}\right) + C\right)$$

guess, $\frac{d}{dt}(te^{5t} - e^{5t})$

$$= 5te^{5t} + e^{5t} - e^{5t}$$

$$= 5te^{5t}$$

$$\frac{d}{dt}(te^{5t} - e^{5t}) = 5e^{5t}t$$

$$\frac{te^{5t} - e^{5t}}{5} = \int e^{5t} t\,dt$$

Initial conditions,

$$x(0) = (1)\left(3\left(\frac{(0)(1) - (1)}{5}\right) + C\right) = 10$$

$$C = 10.6$$

$$x = e^{-5t}\left(3\left(\frac{te^{5t} - e^{5t}}{5}\right) + 10.6\right)$$

$$x = 0.6(t - 1) + 10.6e^{-5t}$$

$$x = 0.6t - 0.6 + 10.6e^{-5t}$$

Figure 6.4 An example of a solution for a first-order system

6.3 INPUT-OUTPUT EQUATIONS

A typical system will be described by more than one differential equation. These equations can be solved to find a single differential equation that can then be integrated. The basic technique is to arrange the equations into an input-output form, such as that in Figure 6.5. These equations will have only a single output variable, and these are always shown on the left hand side. The input variables (there can be more than one) are all on the right hand side of the equation, and act as the non-homogeneous forcing function.

e.g.,

$$2\dddot{y}_1 + \ddot{y}_1 + \dot{y}_1 + 4y_1 = \dot{u}_1 + u_1 + 3u_2 + \ddot{u}_3 + u_3$$

$$\ddot{y}_2 + 6\dot{y}_2 + y_2 = u_1 + 3\ddot{u}_2 + \dot{u}_2 + 0.5u_2 + \dot{u}_3$$

where,

y = outputs
u = inputs

Figure 6.5 Developing input-output equations

A sample derivation of an input-output equation from a system of differential equations is given in Figure 6.6. This begins by replacing the differential operator and combining the equations to eliminate one of the output variables. The solution ends by rearranging the equation to input-output form.

Given the differential equations,

$$\dot{y}_1 = -3y_1 + 2y_2 + u_1 + 2\dot{u}_2 \qquad (1)$$

$$\dot{y}_2 = 2y_1 + y_2 + \dot{u}_1 \qquad (2)$$

Find the input-output equations.

(1) $Dy_1 = -3y_1 + 2y_2 + u_1 + 2Du_2$

$$\therefore y_1(D+3) = 2y_2 + u_1 + 2Du_2$$

$$\therefore y_2 = y_1\left(\frac{D+3}{2}\right) - 0.5u_1 - Du_2$$

(2) $Dy_2 = 2y_1 + y_2 + Du_1$

$$\therefore y_2(D-1) = 2y_1 + Du_1$$

$$\therefore \left(y_1\left(\frac{D+3}{2}\right) - 0.5u_1 - Du_2\right)(D-1) = 2y_1 + Du_1$$

$$\therefore y_1\left(\frac{D^2+2D-3}{2} - 2\right) - 0.5Du_1 + 0.5u_1 - D^2u_2 + Du_2 = Du_1$$

$$\therefore 0.5D^2y_1 + Dy_1 - 3.5y_1 = Du_1 + 0.5Du_1 - 0.5u_1 + D^2u_2 - Du_2$$

$$\therefore 0.5y_1'' + y_1' - 3.5y_1 = u_1' + 0.5u_1' - 0.5u_1 + u_2' - u_2'$$

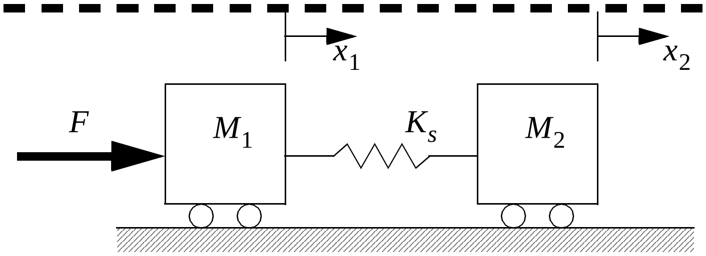
Figure 6.6 An input output equation example

Find the second equation for the example in Figure 6.6 for the output y_2.

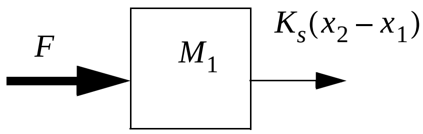
Figure 6.7 Drill problem: Find the second equation in the previous example

6.3.1 Converting Input-Output Equations to State Equations

In some instances we will want to numerically integrate an input-output equation. The example starting in Figure 6.8 shows the development of an input-output equation for two freely rolling masses joined by a spring. The final equation has a derivative on the right hand side that would prevent it from being analyzed in many cases. In particular if the input force 'F' was a step function the first derivative would yield an undefined (infinite) value that could not be integrated.

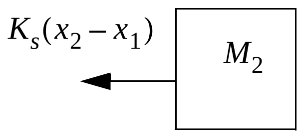


Equations of motion can be derived for these masses.



$$\sum F_x = F + K_s(x_2 - x_1) = M_1 D^2 x_1$$

$$x_1(M_1 D^2 + K_s) = F + K_s x_2$$



$$\sum F_x = -K_s(x_2 - x_1) = M_2 D^2 x_2$$

$$K_s x_1 = x_2(M_2 D^2 + K_s)$$

The equations can be combined to eliminate x2.

$$x_1(M_1 D^2 + K_s) = F + K_s\left(\frac{K_s x_1}{M_2 D^2 + K_s}\right)$$

$$x_1((M_1 D^2 + K_s)(M_2 D^2 + K_s) - K_s^2) = F(M_2 D^2 + K_s)$$

$$x_1(D^4 M_1 M_2 + D^2 K_s(M_1 + M_2) + K_s^2 - K_s^2) = F(M_2 D^2 + K_s)$$

$$x_1(D^4 M_1 M_2 + D^2 K_s(M_1 + M_2)) = F(M_2 D^2 + K_s)$$

$$\left(\frac{d}{dt}\right)^4 x_1 M_1 M_2 + \left(\frac{d}{dt}\right)^2 x_1 K_s(M_1 + M_2) = \left(\frac{d}{dt}\right)^2 F M_2 + F K_s$$

Figure 6.8 Writing an input-output equation as a differential equation

The equation is then converted to state variable form, including a step to calculate a second derivative of the input, as shown in Figure 6.9.

This can then be written in state variable form by creating dummy variables for integrating the function 'F'.

$$\left(\frac{d}{dt}\right)x_1 = v_1$$

$$\left(\frac{d}{dt}\right)v_1 = a_1$$

$$\left(\frac{d}{dt}\right)a_1 = d_1$$

$$\left(\frac{d}{dt}\right)d_1 M_1 M_2 + a_1 K_s(M_1 + M_2) = a_F M_2 + F K_s$$

$$\left(\frac{d}{dt}\right)d_1 = a_1\left(\frac{-K_s(M_1 + M_2)}{M_1 M_2}\right) + a_F\left(\frac{1}{M_1}\right) + F\left(\frac{K_s}{M_1 M_2}\right)$$

The approximate value of the second derivative of a unit time step can be calculated using the time step.

$$a_{F_0} = \frac{1}{T^2} \qquad \text{(when the timestep, step, it is turned on)}$$

$$a_{F_1} = -\frac{1}{T^2} \qquad \text{(after the first timestep)}$$

$$v_F = T a_{F_0} + T a_{F_1} = T\left(\frac{1}{T^2}\right) + T\left(-\frac{1}{T^2}\right) = 0 \qquad \text{(to verify)}$$

$$x_F = \frac{T^2}{2}a_F + \frac{T^2}{2}a_F = \frac{T^2}{2}\left(\frac{1}{T^2}\right) + \frac{T^2}{2}\left(\frac{1}{T^2}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

These equations can then be written in matrix form.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ v_1 \\ a_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-K_s(M_1 + M_2)}{M_1 M_2} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ v_1 \\ a_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{M_1} & \frac{K_s}{M_1 M_2} \end{bmatrix}\begin{bmatrix} a_F \\ F \end{bmatrix}$$

Figure 6.9 Writing state equations for equations with derivatives

6.3.2 Integrating Input-Output Equations

An input-output equation is already in a form suitable for normal integration techniques, with the left hand side being the homogeneous part, and the right hand side is the particular part. If the non homogeneous part includes derivatives, these determine the values of initial conditions. An example of explicitly solving such an equation is shown in Figure 6.10 and Figure 6.11.

The solution begins by evaluating the homogeneous equation.

$$\ddot{x} + 3\dot{x} + 2x = 4\dot{F} + 5F$$

The solution begins by evaluating the homogeneous equation as normal.

$$\ddot{x} + 3\dot{x} + 2x = 0$$

$$A^2 + 3A + 2 = 0$$

$$A = \frac{-3 \pm \sqrt{3^2 - 4(2)}}{2} = -1, -2$$

$$x_h = C_1 e^{-t} + C_2 e^{-2t}$$

The particular solution can also be found as normal, assuming F is unit step function.

Guess, $x = A \qquad \dot{x} = 0 \qquad \ddot{x} = 0$

$$0 + 3(0) + 2A = 2(0) + 5(1) \qquad A = \frac{5}{2}$$

The derivative in the non-homogeneous solution must now be used to find the initial conditions. However the initial position and velocity are known to be zero.

$$\left(\frac{d}{dt}\right)^2 x_0 + 3\left(\frac{d}{dt}\right)x_0 + 2x_0 = 4\left(\frac{d}{dt}F_1\right) + 5F_0$$

$$\left(\frac{d}{dt}\right)^2 x_0 + 3(0) + 2(0) = 4\left(\frac{d}{dt}1\right) + 5(0)$$

$$\left(\frac{d}{dt}\right)^2 x_0 = 4\left(\frac{d}{dt}1\right)$$

$$\frac{d}{dt}x_0 = 4$$

Note: This will be used as an initial condition

for a step function

$$\dot{u}(t) = \frac{1}{dt}$$

Figure 6.10 Integrating an input-output equation

The initial conditions can then be used to find the values of the coefficients. It will be assumed that the system starts undeflected and at rest.

$$x(t) = C_1 e^{-t} + C_2 e^{-2t} + \frac{5}{2}$$

$$x(0) = C_1 e^{-0} + C_2 e^{-0} + \frac{5}{2} = 0 \qquad C_1 = -C_2 - \frac{5}{2}$$

$$\dot{x}(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$$

$$\dot{x}(0) = -C_1 - 2C_2 = 4$$

$$-\left(-C_2 - \frac{5}{2}\right) - 2C_2 = 4 \qquad C_2 = -\frac{3}{2}$$

$$C_1 = -\left(-\frac{3}{2}\right) - \frac{5}{2} = -1$$

The final equation can then be written.

$$x(t) = -e^{-t} - \frac{3}{2}e^{-2t} + \frac{5}{2}$$

Figure 6.11 Integrating an input-output equation (cont'd)

The example in Figure 6.10 is reconsidered with a sinusoidal input in Figure 6.12. In this case the initial acceleration is found to be non-zero. In practical terms, this can be ignored because only the initial position and velocity will be used to find the coefficients.

Assume a sinusoidal input, with the system initially at rest.

$$F(t) = \sin(t) \qquad \dot{F}(t) = \cos(t)$$

$$F(0) = 0 \qquad \dot{F}(0) = 1$$

$$\left(\frac{d}{dt}\right)^2 x_0 + 3\left(\frac{d}{dt}\right)x_0 + 2x_0 = 4(\dot{F}(0)) + 5F(0)$$

$$\left(\frac{d}{dt}\right)^2 x_0 + 3(0) + 2(0) = 4(1) + 5(0)$$

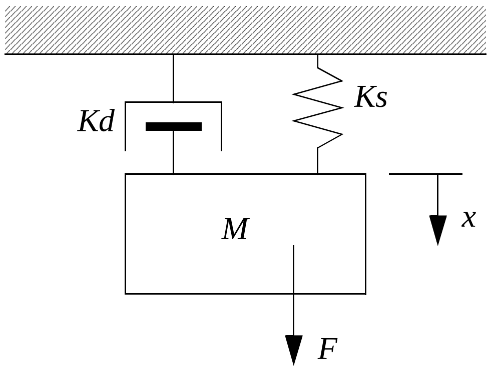
$$\left(\frac{d}{dt}\right)^2 x_0 = 4$$

This indicates that the acceleration will have an initial value, but it will not affect the initial position or velocity.

Figure 6.12 Initial conditions for a sinusoidal input

6.4 DESIGN CASE

The classic mass-spring-damper system is shown in Figure 6.13. In this example the forces are summed to provide an equation. The differential operator is replaced, and the equation is manipulated into transfer function form. The transfer function is given in two different forms because the system is reversible and the output could be either ’F’ or ’x’.



$$\sum F_y = -F + K_d \frac{dx}{dt} + K_s x = -M \frac{d^2x}{dt^2}$$

$$F = M \frac{d^2x}{dt^2} + K_d \frac{dx}{dt} + K_s x$$

$$F = MD^2x + K_d Dx + K_s x$$

$$\frac{F}{x} = MD^2x + K_d Dx + K_s x$$

OR

$$\frac{x}{F} = \frac{1}{MD^2 + K_d D + K_s}$$

Aside: An important concept that is ubiquitous yet largely unrecognized is the use of functional design. We look at parts of systems as self contained modules that use inputs to produce outputs. Some systems (such a mechanisms) are reversible, others are not (consider a internal combustion engine, turning the crank does not produce gasoline). An input is typically something we can change, an output is the resulting change in a system. For the example above 'F' over 'x' implies that we are changing the input 'x', and there is some change in 'F'. We know this could easily be reversed mathematically and practically.

Figure 6.13 A transfer function for a mechanical system

Aside: Keep in mind that the mathematical expression 'F/x' is a ratio between input (displacement action) and output (reaction force). When shown with differentials it is obvious that the ratio is not simple, and is a function of time. Also keep in mind that if we were given a force applied to the system it would become the input (action force) and the output would be the displacement (resulting motion). To do this all we need to do is flip the numerators and denominators in the transfer function.

Mass-spring-damper systems are often used when doing vibration analysis and design work. The first stage of such analysis involves finding the actual displacement for a given displacement or force. A system experiencing a sinusoidal oscillating force is given in Figure 6.14. Numerical values are substituted and the homogeneous solution to the equation is found.

Given the component values input force,

$$M = 1Kg \qquad K_s = 2\frac{N}{m} \qquad K_d = 0.5\frac{Ns}{m} \qquad F = 5\sin(6t)N$$

The differential equation for the mass-spring damper system can be written.

$$1Kg\frac{d^2x}{dt^2} + \left(0.5\frac{Ns}{m}\right)\frac{dx}{dt} + \left(2\frac{N}{m}\right)x = 5\sin(6t)N$$

The homogeneous solution can be determined.

$$1Kg\frac{d^2x}{dt^2} + \left(0.5\frac{Ns}{m}\right)\frac{dx}{dt} + \left(2\frac{N}{m}\right)x = 0$$

$$A = \frac{-0.5\frac{Ns}{m} \pm \sqrt{\left(0.5\frac{Ns}{m}\right)^2 - 4(1Kg)\left(2\frac{N}{m}\right)}}{2(1Kg)}$$

$$A = 0.5(-0.5 \pm \sqrt{0.25 - 8})s^{-1}$$

$$A = (-0.25 \pm 1.392j)s^{-1}$$

$$x_h = C_1 e^{-0.25t}\cos(1.392t + C_2)$$

Figure 6.14 Explicit analysis of a mechanical system

The solution continues in Figure 6.15 where the particular solution is found and put in phase shift form.

The particular solution can now be found with a guess.

$$1Kg\frac{d^2x}{dt^2} + \left(0.5\frac{Ns}{m}\right)\frac{dx}{dt} + \left(2\frac{N}{m}\right)x = 5\sin(6t)N$$

$$x_p = A\sin 6t + B\cos 6t$$

$$x_p' = 6A\cos 6t - 6B\sin 6t$$

$$x_p'' = -36A\sin 6t - 36B\cos 6t$$

$$-36A\sin 6t - 36B\cos 6t + 0.5(6A\cos 6t - 6B\sin 6t) + 2(A\sin 6t + B\cos 6t) = 5\sin(6t)$$

$$-36B + 3A + 2B = 0 \longrightarrow A = \frac{34}{3}B$$

$$-36A - 3B + 2A = 5$$

$$-34\left(\frac{34}{3}B\right) - 3B = 5$$

$$B = \frac{5}{\frac{-34(34)}{3} - 3} = -0.01288 \qquad A = \frac{34}{3}(-0.01288) = -0.1460$$

$$x_p = (-0.1460)\sin 6t + (-0.01288)\cos 6t$$

$$x_p = \frac{\sqrt{(-0.1460)^2 + (-0.01288)^2}}{\sqrt{(-0.1460)^2 + (-0.01288)^2}}((-0.1460)\sin 6t + (-0.01288)\cos 6t)$$

$$x_p = 0.1466(-0.9961\sin 6t - 0.08788\cos 6t)$$

$$x_p = 0.1466\sin\left(6t + \operatorname{atan}\left(\frac{-0.9961}{-0.08788}\right)\right)$$

$$x_p = 0.1466\sin(6t + 1.483)$$

Figure 6.15 Explicit analysis of a mechanical system (continued)

The system is assumed to be at rest initially, and this is used to find the constants in the homogeneous solution in Figure 6.16. Finally the displacement of the mass is used to find the force exerted through the spring on the ground. In this case there are two force frequency components at 1.392rad/s and 6rad/s. The steady-state force at 6rad/s will have a magnitude of .2932N. The transient effects have a time constant of 4 seconds (1/0.25), and should be negligible within a few seconds of starting the machine.

The particular and homogeneous solutions can now be combined.

$$x = x_h + x_p = C_1 e^{-0.25t}\cos(1.392t + C_2) + 0.1466\sin(6t + 1.483)$$

$$x' = -0.25C_1 e^{-0.25t}\cos(1.392t + C_2) - 1.392(C_1 e^{-0.25t}\sin(1.392t + C_2)) + 6(0.1466\cos(6t + 1.483))$$

The initial conditions can be used to find the unknown constants.

$$0 = C_1 e^0 \cos(0 + C_2) + 0.1466\sin(0 + 1.483)$$

$$C_1\cos(C_2) = -0.1460$$

$$C_1 = \frac{-0.1460}{\cos(C_2)}$$

$$0 = -0.25C_1 e^0\cos(0 + C_2) - 1.392(C_1 e^0\sin(0 + C_2)) + 6(0.1466\cos(0 + 1.483))$$

$$0 = -0.25C_1\cos(C_2) - 1.392(C_1\sin(C_2)) + 0.07713$$

$$0 = -0.25\left(\frac{-0.1460}{\cos(C_2)}\cos(C_2)\right) - 1.392\left(\frac{-0.1460}{\cos(C_2)}\sin(C_2)\right) + 0.07713$$

$$0 = 0.0365 + (0.2032)\tan(C_2) + 0.07713$$

$$C_2 = \operatorname{atan}\left(\frac{0.0365 + 0.07713}{-0.2032}\right) = -0.5099$$

$$C_1 = \frac{-0.1460}{\cos(-0.5099)} = -0.1673$$

$$x = (-0.1673e^{-0.25t}\cos(1.392t - 0.5099) + 0.1466\sin(6t + 1.483))m$$

The displacement can then be used to calculate the force transmitted to the ground, assuming the spring is massless.

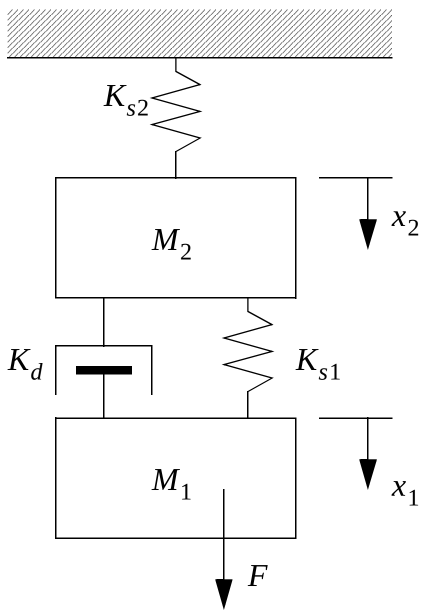
$$F = K_s x$$

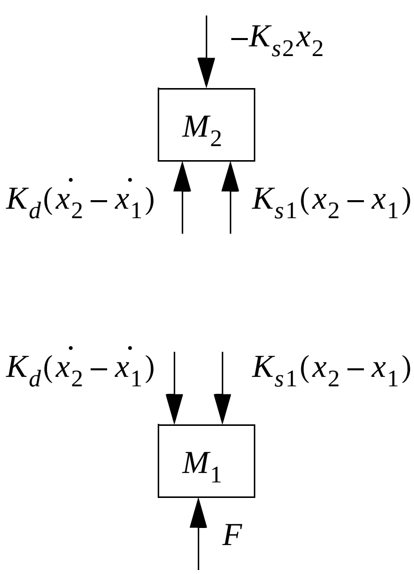
$$F = \left(2\frac{N}{m}\right)(-0.1673e^{-0.25t}\cos(1.392t - 0.5099) + 0.1466\sin(6t + 1.483))m$$

$$F = (-0.3346e^{-0.25t}\cos(1.392t - 0.5099) + 0.2932\sin(6t + 1.483))N$$

Figure 6.16 Explicit analysis of a mechanical system (continued)

A decision has been made to reduce the vibration magnitude transmitted to the ground to 0.1N. This can be done by adding a mass-spring isolator, as shown in Figure 6.17. In the figure the bottom mass-spring-damper combination is the original system. The mass and spring above have been added to reduce the vibration that will reach the ground. Values must be selected for the mass and spring. The design begins by developing the differential equations for both masses.





$$\sum F = -K_{s2}x_2 - K_d(\dot{x}_2 - \dot{x}_1) - K_{s1}(x_2 - x_1) = M_2\ddot{x}_2$$

$$-K_{s2}x_2 - K_d(x_2D - x_1D) - K_{s1}(x_2 - x_1) = M_2x_2D^2$$

$$x_2(-K_{s2} - K_dD - K_{s1} - M_2D^2) = x_1(-K_dD - K_{s1})$$

$$x_1 = x_2\left(\frac{K_{s2} + K_dD + K_{s1} + M_2D^2}{K_dD + K_{s1}}\right) \qquad (1)$$

$$\sum F = K_d(\dot{x}_2 - \dot{x}_1) + K_{s1}(x_2 - x_1) - F = M_1\ddot{x}_1$$

$$K_d(x_2D - x_1D) + K_{s1}(x_2 - x_1) - F = M_1x_1D^2$$

$$x_1(-K_dD - K_{s1} - M_1D^2) + x_2(K_dD + K_{s1}) = F \qquad (2)$$

Figure 6.17 Vibration isolation system

For the design we are only interested in the upper spring, as it determines the force on the ground. An input-output equation for that spring is developed in Figure 6.18. The given values for the mass-spring-damper system are used. In addition a value for the upper mass is selected. This is arbitrarily chosen to be the same as the lower mass. This choice may need to be changed later if the resulting spring constant is not practical.

The solution begins by combining equations (1) and (2) and inserting the.numerical values for the lower mass, spring and damper. We can also limit the problem by selecting a mass value for the upper mass.

$$x_2\left(\frac{K_{s2} + K_d D + K_{s1} + M_2 D^2}{K_d D + K_{s1}}\right)(-K_d D - K_{s1} - M_1 D^2) + x_2(K_d D + K_{s1}) = F$$

$$M_1 = 1Kg \qquad K_{s1} = 2\frac{N}{m} \qquad K_d = 0.5\frac{Ns}{m} \qquad M_2 = 1Kg$$

$$x_2\left(\frac{K_{s2} + 0.5D + 2 + D^2}{0.5D + 2}\right)(-0.5D - 2 - D^2) + x_2(0.5D + 2) = F$$

$$x_2(D^2 + 0.5D + 2 + K_{s2})(D^2 - 0.5D - 2) + x_2(0.5D + 2)^2 = F(0.5D + 2)$$

$$x_2(D^4(-1) + D^2(K_{s2}) + D^1(-0.5K_{s2}) + (-2K_{s2})) = F(0.5D + 2)$$

This can now be converted back to a differential equation and combined with the force.

$$-\left(\frac{d}{dt}\right)^4 x_2 + K_{s2}\left(\frac{d}{dt}\right)^2 x_2 - 0.5K_{s2}\left(\frac{d}{dt}\right)x_2 - 2K_{s2}x_2 = 0.5\left(\frac{d}{dt}\right)F + 2F$$

$$-\left(\frac{d}{dt}\right)^4 x_2 + K_{s2}\left(\frac{d}{dt}\right)^2 x_2 - 0.5K_{s2}\left(\frac{d}{dt}\right)x_2 - 2K_{s2}x_2 = 0.5\left(\frac{d}{dt}\right)5\sin(6t) + 2(5)\sin(6t)$$

$$-\left(\frac{d}{dt}\right)^4 x_2 + K_{s2}\left(\frac{d}{dt}\right)^2 x_2 - 0.5K_{s2}\left(\frac{d}{dt}\right)x_2 - 2K_{s2}x_2 = 15\cos(6t) + 10\sin(6t)$$

Figure 6.18 Developing an input output equation

This particular solution of the differential equation will yield the steady-state displacement of the upper mass. This can then be used to find the needed spring coefficient.

The particular solution begins with a guess.

$$-\left(\frac{d}{dt}\right)^4 x_2 + K_{s2}\left(\frac{d}{dt}\right)^2 x_2 - 0.5K_{s2}\left(\frac{d}{dt}\right)x_2 - 2K_{s2}x_2 = 15\cos(6t) + 10\sin(6t)$$

$$x_p = A\sin 6t + B\cos 6t$$

$$\left(\frac{d}{dt}\right)x_p = 6A\cos 6t - 6B\sin 6t$$

$$\left(\frac{d}{dt}\right)^2 x_p = -36A\sin 6t - 36B\cos 6t$$

$$\left(\frac{d}{dt}\right)^3 x_p = -216A\cos 6t + 216B\sin 6t$$

$$\left(\frac{d}{dt}\right)^4 x_p = 1296A\sin 6t + 1296B\cos 6t$$

$$\sin(6t)(-1296A - 36AK_{s2} + 0.5K_{s2}6B - 2K_{s2}A) = 10\sin(6t)$$

$$A(-1296 - 38K_{s2}) + B(3K_{s2}) = 10$$

$$B = \frac{10 + A(1296 + 38K_{s2})}{3K_{s2}}$$

$$\cos(6t)(-1296B - 36BK_{s2} + (-0.5)K_{s2}6A - 2K_{s2}B) = 15\cos(6t)$$

$$A(-3K_{s2}) + B(-1296 - 38K_{s2}) = 15$$

$$A(-3K_{s2}) + \frac{10 + A(1296 + 38K_{s2})}{3K_{s2}}(-1296 - 38K_{s2}) = 15$$

$$A(-9K_{s2}^2) + (10 + A(1296 + 38K_{s2}))(-1296 - 38K_{s2}) = 45K_{s2}$$

$$A\left(\frac{-9K_{s2}^2}{(-1296 - 38K_{s2})}\right) + A(1296 + 38K_{s2}) = \frac{45K_{s2}}{(-1296 - 38K_{s2})} - 10$$

$$A = \frac{\dfrac{45K_{s2}}{(-1296 - 38K_{s2})} - 10}{\dfrac{-9K_{s2}^2}{(-1296 - 38K_{s2})} + (1296 + 38K_{s2})}$$

$$A = \frac{45K_{s2} + 10(1296 + 38K_{s2})}{-9K_{s2}^2 + (1296 + 38K_{s2})(-1296 - 38K_{s2})}$$

$$A = \frac{425K_{2s} + 12960}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}$$

Figure 6.19 Finding the particular solution

The value for B can then be found.

$$B = \frac{10 + \left(\frac{425K_{2s} + 12960}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}\right)(1296 + 38K_{s2})}{3K_{s2}}$$

$$B = \frac{10(-1453K_{2s}^2 - 98496K_{2s} - 1679616) + (425K_{2s} + 12960)(1296 + 38K_{s2})}{3K_{s2}(-1453K_{2s}^2 - 98496K_{2s} - 1679616)}$$

$$B = \frac{1620K_{2s}^2 + 2.097563\times10^8 K_{2s}}{3K_{s2}(-1453K_{2s}^2 - 98496K_{2s} - 1679616)}$$

$$B = \frac{540K_{2s} + 69918768}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}$$

Figure 6.20 Finding the particular solution (cont'd)

Finally the magnitude of the particular solution is calculated and set to the desired amplitude of 0.1N. This is then used to calculate the spring coefficient.

$$amplitude = \sqrt{A^2 + B^2}$$

$$0.1 = \sqrt{\left(\frac{425K_{2s} + 12960}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}\right)^2 + \left(\frac{540K_{2s} + 69918768}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}\right)^2}$$

A value for the spring coefficient was then found using Mathcad to get a value of 662N/m.

$K := 1$

given

$$\sqrt{\left[\frac{425\cdot K + 12960}{(-1453\cdot K\cdot K) - 98496\cdot K - 1679616}\right]^2 + \left[\frac{540\cdot K + 69918768}{(-1453\cdot K\cdot K) - 98496\cdot K - 1679616}\right]^2} = 0.1$$

$\text{find}(K) = 661.68$

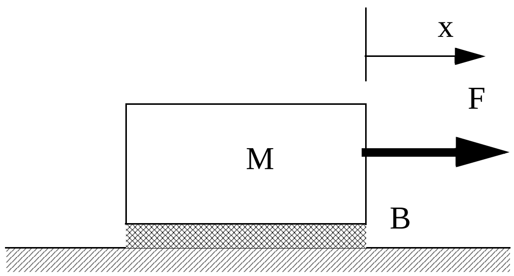
Figure 6.21 Calculation of the spring coefficient

6.5 SUMMARY

- The differential operator can be manipulated algebraically
- Equations can be manipulated into input-output forms and solved as normal differential equations

6.6 PRACTICE PROBLEMS

1. Develop the input-output equation for the mechanical system below. There is viscous damping between the block and the ground. A force is applied to cause the mass the accelerate.



2. Find the input-output form for the following equations.

$$\dot{y} + y + x = 3$$

$$\dot{x} + x + y = 0$$

3. Find the input-output form for the following equations.

$$\ddot{x}_1 + \dot{x}_1 + 2x_1 - \dot{x}_2 - x_2 = 0$$

$$-\dot{x}_1 - x_1 + \ddot{x}_2 + \dot{x}_2 + x_2 = F$$

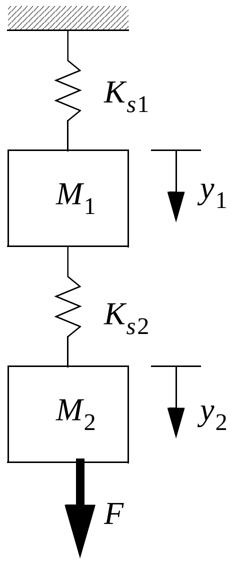
4. The following differential equations were converted to the matrix form shown. Use Cramer's rule to find an input-output equation for 'y'.

$$\ddot{y} + 2x = \frac{F}{10}$$

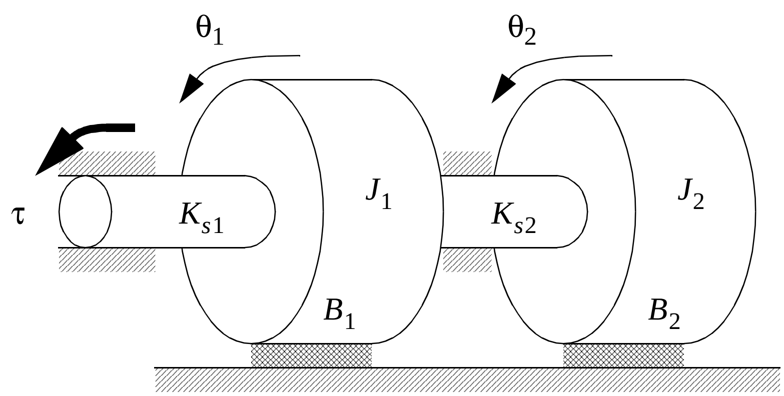
$$7\dot{y} + 4y + 9\ddot{x} + 3x = 0$$

$$\begin{bmatrix} (D^2) & (2) \\ (7D+4) & (9D^2+3) \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} \frac{F}{10} \\ 0 \end{bmatrix}$$

5. Find the input output equation for y2. Ignore the effects of gravity.

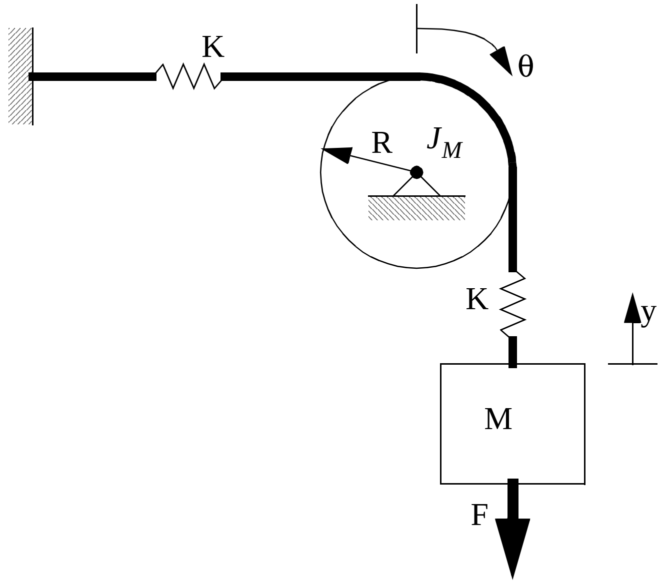


6. Find the input-output equations for the systems below. Here the input is the torque on the left hand side.

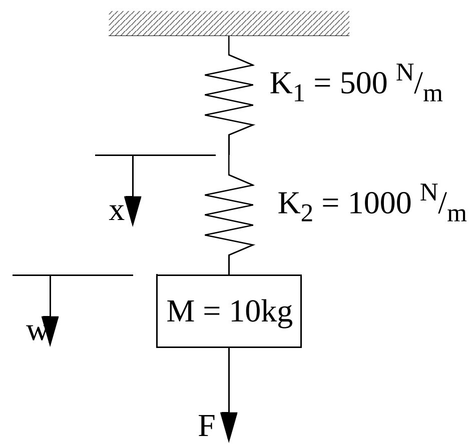


7. Write the input-output equations for the mechanical system below. The input is force ‘F’, and the output is ‘y’ or the angle theta (give both equations). Include the inertia of both masses, and

gravity for mass ‘M’.



8. The applied force ‘F’ is the input to the system, and the output is the displacement ‘x’.



a) Find x(t), given F(t) = 10N for t >= 0 seconds.
b) Using numerical methods, find the steady-state response for an applied force of F(t) = 10cos(t + 1) N ?
a) Solve the differential equation to find the explicit response for an applied force of F(t) = 10cos(t + 1) N ?
d) Set the acceleration to zero and find an approximate solution for an applied force of F(t) = 10cos(t + 1) N. Compare the solution to the previous solutions.

6.7 PRACTICE PROBLEM SOLUTIONS

1.

$$\ddot{x} + \dot{x}\left(\frac{B}{M}\right) = \frac{F}{M}$$

2.

$$\ddot{x} + 2\dot{x} = -3$$
$$\ddot{y} + 2\dot{y} = 3$$

3.

$$\left(\frac{d}{dt}\right)^4 x_1 + 2\left(\frac{d}{dt}\right)^3 x_1 + 3\left(\frac{d}{dt}\right)^2 x_1 + \left(\frac{d}{dt}\right) x_1 + x_1 = \left(\frac{d}{dt}\right) F + F$$
$$\left(\frac{d}{dt}\right)^4 x_2 + 2\left(\frac{d}{dt}\right)^3 x_2 + 3\left(\frac{d}{dt}\right)^2 x_2 + \left(\frac{d}{dt}\right) x_2 + x_2 = \left(\frac{d}{dt}\right)^2 F + \left(\frac{d}{dt}\right) F + 2F$$

4.

$$y = \frac{\left|\begin{bmatrix} \frac{F}{10} & (2) \\ 0 & (9D^2+3) \end{bmatrix}\right|}{\left|\begin{bmatrix} (D^2) & (2) \\ (7D+4) & (9D^2+3) \end{bmatrix}\right|} = \frac{\frac{F}{10}(9D^2+3)}{D^2(9D^2+3) - 2(7D+4)} = \frac{F(0.9D^2+0.3)}{9D^4 + 3D^2 - 14D - 8}$$

$$y(9D^4 + 3D^2 - 14D - 8) = F(0.9D^2 + 0.3)$$

$$\left(\frac{d}{dt}\right)^4 y(9) + \left(\frac{d}{dt}\right)^2 y(3) + \left(\frac{d}{dt}\right)^1 y(-14) + y(-8) = \left(\frac{d}{dt}\right)^2 F(0.9) + F(0.3)$$

$$\left(\frac{d}{dt}\right)^4 y + \left(\frac{d}{dt}\right)^2 y\left(\frac{1}{3}\right) + \left(\frac{d}{dt}\right)^1 y\left(\frac{-14}{9}\right) + y\left(\frac{-8}{9}\right) = \left(\frac{d}{dt}\right)^2 F\left(\frac{1}{10}\right) + F\left(\frac{1}{3}\right)$$

5.

$$\left(\frac{d}{dt}\right)^4 y_2 + \left(\frac{d}{dt}\right)^2 y_2\left(\frac{K_{s1}M_2 + K_{s2}M_2 + K_{s2}M_1}{M_1M_2}\right) + y_2\left(\frac{K_{s1}K_{s2}}{M_1M_2}\right) =$$
$$\left(\frac{d}{dt}\right)^2 F\left(\frac{M_1}{M_1M_2}\right) + F\left(\frac{K_{s1} + K_{s2}}{M_1M_2}\right)$$

6.

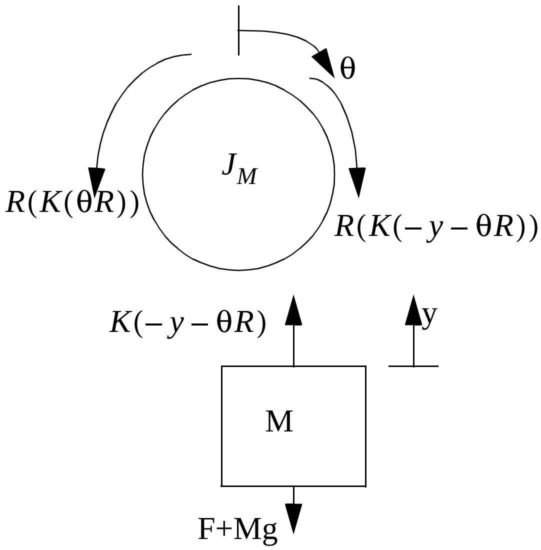
$$\left(\frac{d}{dt}\right)^4 \theta_2 + \left(\frac{d}{dt}\right)^3 \theta_2\left(\frac{J_1 B_2 + J_2 B_1}{J_1 J_2}\right) + \left(\frac{d}{dt}\right)^2 \theta_2\left(\frac{J_1 K_{s2} + J_2 K_{s2} + B_1 B_2}{J_1 J_2}\right) +$$

$$\left(\frac{d}{dt}\right)\theta_2\left(\frac{B_1 K_{s2} + B_2 K_{s2}}{J_1 J_2}\right) = \tau\frac{K_{s2}}{J_1 J_2}$$

$$\left(\frac{d}{dt}\right)^4 \theta_1 + \left(\frac{d}{dt}\right)^3 \theta_1\left(\frac{J_1 B_2 + J_2 B_1}{J_1 J_2}\right) + \left(\frac{d}{dt}\right)^2 \theta_1\left(\frac{J_1 K_{s2} + J_2 K_{s2} + B_1 B_2}{J_1 J_2}\right) +$$

$$\left(\frac{d}{dt}\right)\theta_1\left(\frac{B_1 K_{s2} + B_2 K_{s2}}{J_1 J_2}\right) = \left(\frac{d}{dt}\right)^2 \tau\left(\frac{1}{J_1}\right) + \left(\frac{d}{dt}\right)\tau\left(\frac{B_2}{J_1 J_2}\right) + \tau\left(\frac{K_{s2}}{J_1 J_2}\right)$$

7.



$$\sum M = -R(K(\theta R)) + R(K(-y-\theta R)) = J_M\ddot{\theta}$$

$$R^2K\theta + RKy + R^2K\theta = -J_M\ddot{\theta}$$

$$\theta\left(\frac{2R^2K + J_MD^2}{-RK}\right) = y$$

$$\sum F = K(-y-\theta R) - F - Mg = M\ddot{y}$$

$$K(y+\theta R) + F + Mg = -MyD^2$$

$$\theta(KR) + F + Mg = y(-MD^2 - K)$$

$$\theta(KR) + y(MD^2 + K) = -F - Mg$$

for the theta output equation;

$$\theta(KR) + \theta\left(\frac{2R^2K + J_MD^2}{-RK}\right)(MD^2 + K) = -F - Mg$$

$$\theta(-K^2R^2) + \theta(2R^2K + J_MD^2)(MD^2 + K) = FKR + MgKR$$

$$\theta(-K^2R^2 + 2R^2MKD^2 + 2R^2K^2 + J_MMD^4 + J_MKD^2) = FKR + MgKR$$

$$\left(\frac{d}{dt}\right)^4\theta\left(\frac{J_MM}{KR}\right) + \left(\frac{d}{dt}\right)^4\theta(2R^2M + J_M) + \theta(K^2R^2) = FKR + MgKR$$

$$\left(\frac{d}{dt}\right)^4\theta + \left(\frac{d}{dt}\right)^4\theta\left(\frac{2R^3K}{J_M} + \frac{KR}{M}\right) + \theta\left(\frac{K^3R^3}{J_MM}\right) = F\frac{K^2R^2}{J_MM} + \frac{MgK^2R^2}{J_MM}$$

for the y output equation;

$$y\left(\frac{-RK}{2R^2K + J_MD^2}\right)(KR) + y(MD^2 + K) = -F - Mg$$

$$y\left(2R^2MD^2 + \frac{J_MD^4M}{K} + K2R^2 + \frac{J_MD^2}{K} - R^2K\right) =$$

$$-F\left(2R^2 + \frac{J_MD^2}{K}\right) - Mg\left(2R^2 + \frac{J_MD^2}{K}\right)$$

$$\left(\frac{d}{dt}\right)^4 y\left(\frac{J_MM}{K}\right) + \left(\frac{d}{dt}\right)^2 y\left(2R^2M + \frac{J_M}{K}\right) + y(R^2K) =$$

$$\left(\frac{d}{dt}\right)^2 F\left(\frac{-J_M}{K}\right) + F(-2R^2) + (-2MgR^2)$$

$$\left(\frac{d}{dt}\right)^4 y + \left(\frac{d}{dt}\right)^2 y\left(\frac{2KR^2}{J_M} + \frac{1}{M}\right) + y\left(\frac{R^2K^2}{J_MM}\right) =$$

$$\left(\frac{d}{dt}\right)^2 F\left(\frac{-1}{M}\right) + F\left(\frac{-2KR^2}{J_MM}\right) + \left(\frac{-2gKR^2}{J_M}\right)$$

8.

$$\ddot{x} + x\left(\frac{K_1K_2}{M(K_1+K_2)}\right) = F\left(\frac{K_2}{M(K_1+K_2)}\right) + g\left(\frac{K_2}{K_1+K_2}\right)$$

a) $x(t) = -0.2168\cos(5.774t) + 0.2162$

b) $x(t) = -0.2168\cos(5.774t) + 0.02\cos(t+1) + 0.1962$

c) $x(t) = 0.02\cos(t+1) + 0.1962$

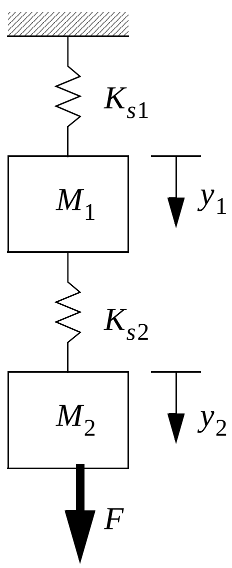
6.8 ASSGINMENT PROBLEMS

1. Find the input-output equations for the differential equations below if both 'x' and 'y' are outputs.

$$\dot{y} + y + 5x = 3$$

$$\dot{x} + x + y = 7$$

2. For the system pictured below find the output response as a function of time for y1 and y2 using a) integration to find an explicit function, b) numerical analysis using Scilab or C.



$$K_{s1} = 100\frac{N}{m}$$

$$M_1 = 1Kg$$

$$K_{s2} = 1000\frac{N}{m}$$

$$M_2 = 0.1Kg$$

6.9 REFERENCES

Irwin, J.D., and Graf, E.R., Industrial Noise and Vibration Control, Prentice Hall Publishers, 1979.

Close, C.M. and Frederick, D.K., “Modeling and Analysis of Dynamic Systems, second edition, John Wiley and Sons, Inc., 1995.