21. STATE SPACE CONTROLLERS

Topics:

Objectives:

21.1 INTRODUCTION

• There are a few terms to be defined,

identification - the process of determining a state space (or other) model of a process.
parameter identification - used to estimate model parameter values.
estimatior - this tries to determine state variable values given available data.
observer - a special type of estimator that uses available input and output data to estimate a system state given unknown initial conditions. This is simalar to running a system simulation in parallel with the real system.
compensator - a feedback control system designed to drive a system to an arbitrary input.
regulator - a feedback control system designed to drive a system to a given operating point. A special case of a compensator with a zero input.

21.2 FULL STATE FEEDBACK

- The basic definition of a feedback controller is below,

The system state equations, with D=0,

$$\dot{x} = Ax + Bu$$

$$\dot{y} = Cx + 0u$$

The Eigenvalues of A give the system poles

$$u = r - Kx$$

r = setpoint

u = control output/plant input

K = proportional gain values

$$\dot{x} = Ax + B(r - Kx)$$

$$\dot{x} = (A - BK)x + Br$$

$$\dot{x} = A_c x + Br$$

$$\dot{y} = Cx$$

- An example of a controller design follows, The controller design involves calculating K values to meet control objectives.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u \qquad \dot{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x$$

Check for controlability,

$$M_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & -12 \end{bmatrix}$$

The rank is 2, and the order of the matrix A is 2, so the system is controllable.

Check for the stability of the system without a controller,

$$|sI - A| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right| = \begin{vmatrix} s & -1 \\ -2 & s-3 \end{vmatrix}$$

$$|sI - A| = s(s-3) - 2 = s^2 - 3s - 2 = (s - \quad)(s - \quad)$$

Therefore, the uncompensated system is.......

Select an input vector with variables,

$$u = -Kx = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} x$$

$$A_c = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 4k_1 & 4k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ (-2-4k_1) & (-3-4k_2) \end{bmatrix}$$

$$|sI - A_c| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ (-2-4k_1) & (-3-4k_2) \end{bmatrix} \right| = \begin{vmatrix} s & -1 \\ s+2+4k_1 & s+3+4k_2 \end{vmatrix}$$

$$|sI - A_c| = s(s+3+4k_2) + (s+2+4k_1) = s^2 + s(4+4k_2) + (2+4k_1)$$

The system is clearly second order, so the second order methods may be used to select parameters. However for variety, why don't we select a system that has poles at,

$$p = 1+2j, 1-2j$$

$$s^2 + s(4+4k_2) + (2+4k_1) = (s+1+2j)(s+1-2j)$$

$$s^2 + s(4+4k_2) + (2+4k_1) = s^2 + s(1+2j+1-2j) + ((1+2j)(1-2j))$$

$$s^2 + s(4+4k_2) + (2+4k_1) = s^2 + s(2) + (5)$$

$$4 + 4k_2 = 2 \qquad k_2 = -0.5$$

$$2 + 4k_1 = 5 \qquad k_1 = 0.75$$

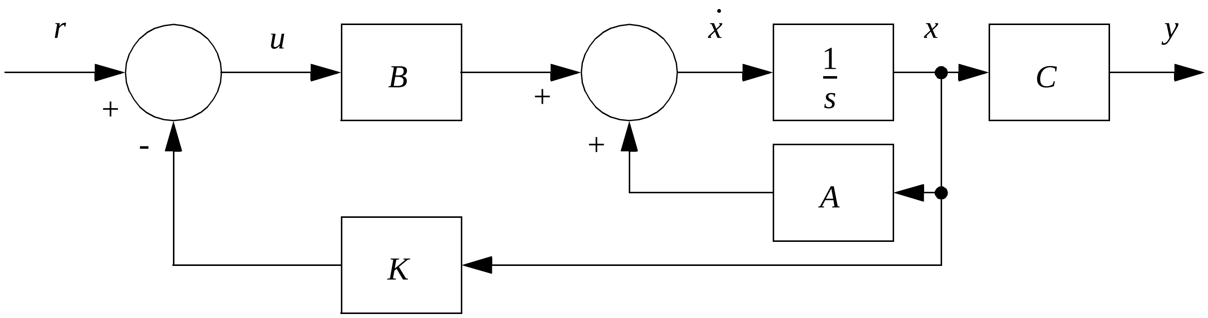


Figure 21.1 Block diagram of a state based controller

• The value of K in the previous example could also be picked using Ackermann's formula.

$$K = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix} M_c^{-1} \Phi_d(A)$$

where,

$$\Phi_d(s) = \text{the desired response}$$

• Consider the previous example,

The state equations and the desired response of the system are,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u \qquad \dot{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x$$

$$\Phi_d(s) = s^2 + s(2) + (5)$$

The Markov (?) parameter can be calculated first.

$$M_c = \left[\begin{bmatrix} 0 \\ 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right] = \begin{bmatrix} 0 & 4 \\ 4 & -12 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 4 & -12 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^2 + \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} (2) + (5) I \right)$$

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} -12 & -4 \\ -4 & 0 \end{bmatrix}}{0 - 16} \left(\begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -4 & -6 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right)$$

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -0.75 & -0.25 \\ -0.25 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} -0.25 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} -0.75 & 0.25 \end{bmatrix}$$

NOTE: DOESN'T MATCH, TRACK DOWN ALGEBRA PROB

21.3 OBSERVERS

• Observers are used to estimate the next system state. They can also be used to predict future state values.

• As an example, an observer might be used for a targeting system that is tracking a moving object. The estimator can be used to direct a missle to an estimated location so that the missle and target arrive at the same time.

• An observer is shown in block diagram form

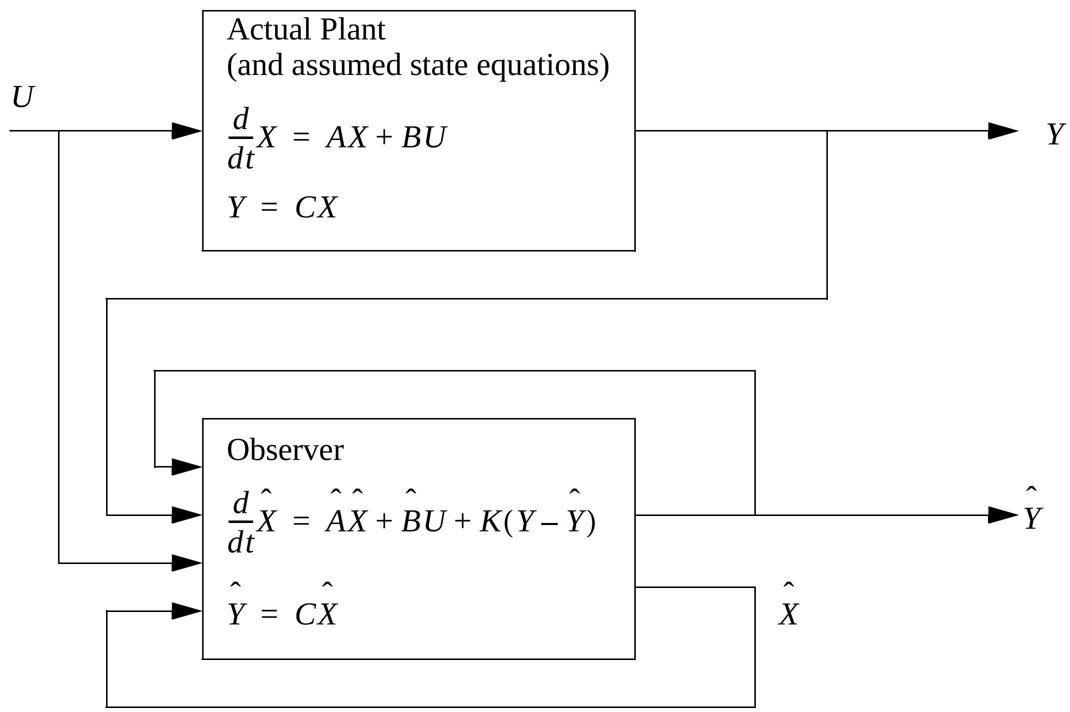


Figure 21.2 Block diagram of an observer

Aside: For an open loop estimator a simulation is run, while the system is running. The simulation values are compared to the actual system responses, and the model is adjusted. It is assumed that the initial value of the state is unknown. If A is stable then the estimation will eventually converge (this can be checked using the Eigenvalues). However, there is no guarantee that it will converge quickly.

The actual system

$$\frac{d}{dt}x = Ax + Bu \qquad y = Cx \qquad C \neq I$$

The state estimator model

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu$$

The estimation error

$$\tilde{x} = x - \hat{x}$$

$$\frac{d}{dt}(x - \hat{x}) = A(x - \hat{x}) \qquad \frac{d}{dt}\tilde{x} = A\tilde{x}$$

$$\tilde{x}(t) = e^{At}\tilde{x}(0)$$

• A closed loop estimatior is shown below. It uses a learning matrix L to adjust the convergence rate. A larger L value will result in faster convergence, but the exponent must now be checked to ensure observability.

The system and estimator

$$y = Cx \qquad \hat{y} = C\hat{x}$$

The estimator error

$$\tilde{y} = y - \hat{y} = C\tilde{x}$$

The estimator

$$\frac{d}{dt}\hat{x} = A\hat{x}(t) + Bu(t) + L(y(t) - \tilde{y}(t))$$

$$\hat{y}(t) = C\hat{x}(t)$$

where,

$$L = \text{a gain matrix}$$

The convergence rate,

$$\frac{d}{dt}\tilde{x} = (A - LC)\tilde{x}$$

$$\tilde{x}(t) = e^{(A - LC)t}\tilde{x}(0)$$

- Estimator gain selection

If a system is controllable the Eigenvalues of A-BK can be placed anywhere,

$$rank(A) = rank\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n$$

If a system is observable the Eigenvalues of A-LC can be placed anywhere,

$$rank(A) = rank\begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix} = n$$

- Consider the example below, for a second order system. The learning rate values are selected to determine how quickly the model converges.

Consider the state equations,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the convergence rate and then the Eigenvalues,

$$A - LC = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -a_1 - l_1 & 1 \\ -a_2 - l_2 & 0 \end{bmatrix}$$

$$|sI - (A - LC)| = \begin{vmatrix} s + a_1 + l_1 & -1 \\ a_2 + l_2 & s \end{vmatrix} = s(s + a_1 + l_1) - (-1)(a_2 + l_2)$$

$$= s^2(1) + s(a_1 + l_1) + (a_2 + l_2)$$

• This can also be done with Ackermann's formula,

$$L = K_e^T = \Phi_e(s) M_o^{-1}\begin{bmatrix} 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}$$

Consider the state equations for a mass spring damper system (y is the position).

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}F \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}F$$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

First test for observability, (note: it is observable)

$$rank\begin{bmatrix} C \\ CA \end{bmatrix} = rank\begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix} \end{bmatrix} = rank\begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} = rank\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2$$

Consider the open loop estimator

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu \qquad \hat{y} = C\hat{x}$$

$$\frac{d}{dt}\begin{bmatrix} \hat{x} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{v} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}F \qquad \hat{y} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{v} \end{bmatrix}$$

• The open loop estimator will normally ignore the response to initial conditions and will have long term errors resulting from modeling errors and random disturbances in the system.

Consider the closed loop estimator with a learning rate.

$$\frac{d}{dt}\hat{x} = A\hat{x}(t) + Bu(t) + L\tilde{y}(t) \qquad \hat{y}(t) = C\hat{x}(t)$$

$$\frac{d}{dt}\begin{bmatrix} \hat{x} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{v} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}F + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\tilde{y} \qquad \hat{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{v} \end{bmatrix}$$

These are then simulated numerically using initial conditions of zero.

XXXXXXXXXXX Example of learning parameter estimator to obtain desired responses.

21.4 SUPPLEMENTAL OBSERVERS

• In many cases some state variables can be measured directly, but others cannot. In these cases it is more efficient to only estimate the unmeasurable variable values [Eronini].

• This can be

$$W = KX$$

$$X = \begin{bmatrix} C \\ \overline{K} \end{bmatrix}^{-1} \begin{bmatrix} Y \\ \overline{W} \end{bmatrix} = M^{-1} \begin{bmatrix} Y \\ \overline{W} \end{bmatrix} \qquad M = \begin{bmatrix} C \\ \overline{K} \end{bmatrix} \quad \text{(Note: the inverse must exist)}$$

$$\begin{bmatrix} Y \\ \overline{W} \end{bmatrix} = MX$$

$$\frac{d}{dt}\begin{bmatrix} Y \\ \overline{W} \end{bmatrix} = \frac{d}{dt}MX = M(AX + BU) = MAX + MBU = MAM^{-1}\begin{bmatrix} Y \\ \overline{W} \end{bmatrix} + MBU$$

The matrices can be partitioned to isolate the unmeasurable state variables. Note that the Eigenvalues of Hw must be relatively small.

$$MAM^{-1} = \begin{bmatrix} \dots & \dots \\ H_y & H_w \end{bmatrix} \qquad MB = \begin{bmatrix} \dots \\ B_w \end{bmatrix}$$

$$\frac{d}{dt}\hat{W} = H_y Y + H_w \hat{W} + B_w U$$

21.5 REGULATED CONTROL WITH OBSERVERS

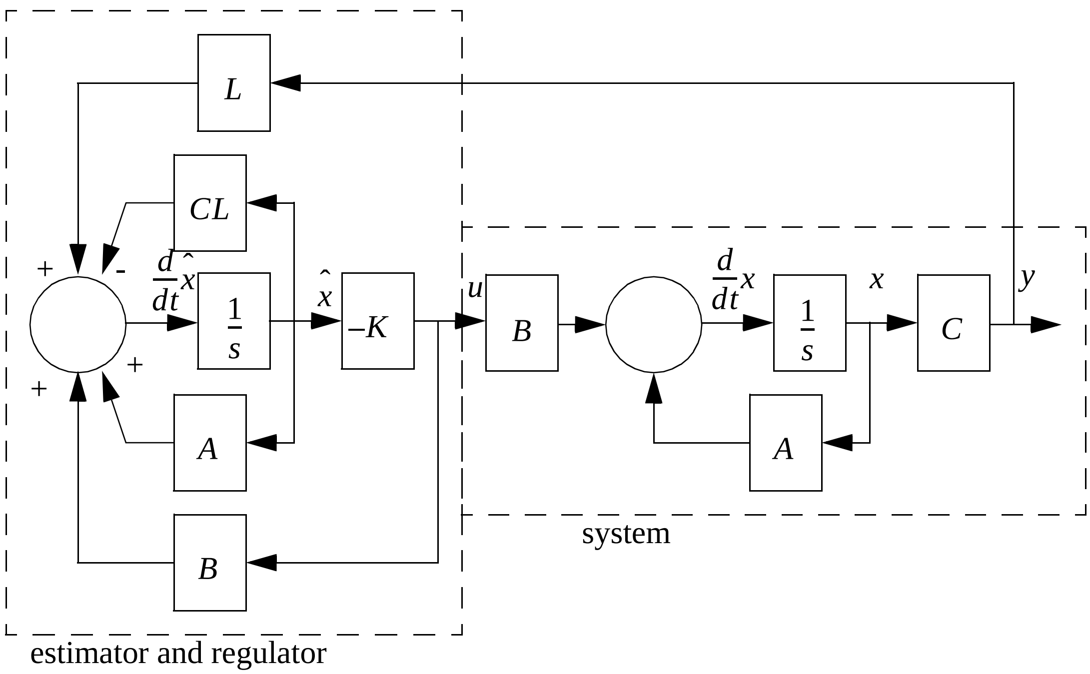
• For a regulated system the control output ’u’ can be estimated using the desired system state.

$$u = Kx = \hat{K}y$$

• The system state is often difficult to measure directly, so an estimator can be used. .

$$u = -Kx = -K\hat{x}$$

• A regulator can use an estimate of the system state to



For a regulator assume the state variables go to zero, so....

$$u = -K\hat{x}$$

A closed loop estimator

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + L(y - \hat{y})$$

$$\hat{y} = C\hat{x}$$

The state model for the system is,

$$\frac{d}{dt}x = Ax + Bu$$

$$y = Cx$$

These can be combined to obtain,

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + L(y - \hat{y}) = \hat{x}(A - BK - LC) + x(LC)$$

$$\frac{d}{dt}x = Ax + Bu = x(A) + \hat{x}(-BK)$$

Resulting in the matrix,

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

An error variable can be introduced, and written in matrix form,

$$\tilde{x} = x - \hat{x}$$

$$\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = Tx_{cl} \approx \tilde{x}_{cl} \qquad T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = T^{-1}$$

An error variable can be introduced, and written in matrix form,

$$A_{cl} = TA_{cl}T^{-1} \approx \overline{A_{cl}}$$

The previous matrix can be revised

$$\overline{A_{cl}} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$$

$$\overline{A_{cl}} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A-BK & BK \\ A-BK & -A+BK+LC \end{bmatrix}$$

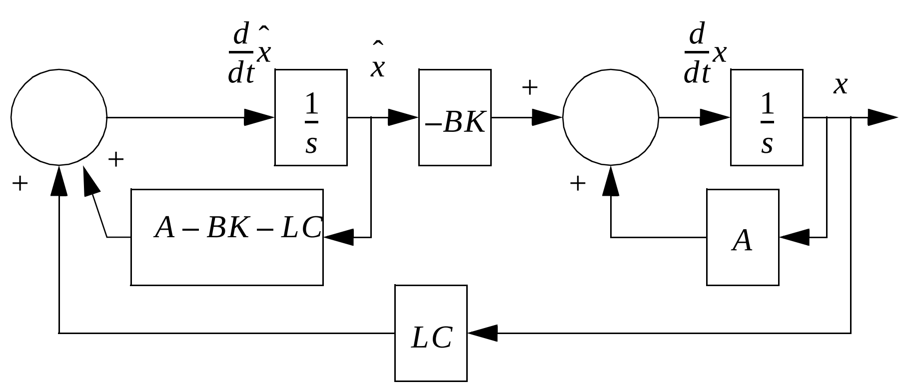
$$\overline{A_{cl}} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}$$

The new poles for this system can be written,

$$\left|sI - \overline{A_{cl}}\right| = (A-BK)(A-LC) - BK(0) = A^2 - ALC - ABK + BKLC$$

• The estimators and regulators can be designed separately and then combined. This is the separation principle.

• The simplified block diagram for the system is,



• The compensator,

The previous estimator equations can be combined with a compensator

$$u = -K\hat{x}$$

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + L(y - \hat{y}) = A\hat{x} - BK\hat{x} + L(y - C\hat{x}) = (A - BK - LC)\hat{x} + (L)y$$

These can be used to define a new compenator state,

$$\hat{x} = x_c$$

$$u = -C_c x_c$$

$$\frac{d}{dt}x_c = A_c x_c + B_c y$$

where,

$$A_c = A - BK - LC$$
$$B_c = L$$
$$C_c = K$$

• This can be converted to a transfer function for the regulator.

These can be used to define a new compenator state,

$$sx_c = A_c x_c + B_c y$$
$$x_c(sI - A_c) = B_c y$$
$$\frac{x_c}{y} = \frac{B_c}{sI - A_c}$$

$$u = -C_c x_c$$
$$\frac{u}{x_c} = -C_c$$
$$\frac{u}{y} = \frac{u}{x_c}\frac{x_c}{y} = \frac{-C_c B_c}{sI - A_c}$$

• The system can also be set up for a compensator by assuming the setpoint input is not zero, First a system error is defined.

$$e = r - y$$

The previously defined compensator can be used here,

$$e = r - y$$

$$u = -C_c x_c$$

$$\frac{u}{x_c} = -C_c$$

$$\frac{d}{dt}x_c = A_c x_c + B_c e$$

$$\frac{x_c}{e} = \frac{B_c}{sI - A_c}$$

$$\frac{u}{e} = \frac{u}{x_c}\frac{x_c}{e} = \frac{-C_c B_c}{sI - A_c}$$

The system can also be modelled,

$$y = Cx$$

$$\frac{y}{x} = C$$

$$\frac{d}{dt}x = Ax + Bu$$

$$\frac{x}{u} = \frac{B}{sI - A}$$

$$\frac{y}{u} = \frac{y}{x}\frac{x}{u} = \frac{CB}{sI - A}$$

The feedforward and feedback loop can be simplified,

$$\frac{y}{e} = \frac{u}{e}\frac{y}{u} = \frac{-C_c B_c CB}{(sI - A_c)(sI - A)}$$

$$\frac{y}{r} = \frac{\frac{-C_c B_c CB}{(sI - A_c)(sI - A)}}{1 + \frac{-C_c B_c CB}{(sI - A_c)(sI - A)}} = \frac{-C_c B_c CB}{(sI - A_c)(sI - A) - C_c B_c CB}$$

• Consider an example of a regulator/estimator design, Consider the state equations for a mass-spring-damper system.

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}F \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}F$$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

The regulator will be designed to have a response time of 0.5s and damped frequency of 2 rad/s.

$$s = \frac{1}{0.5} + j(2) = -2 \pm 2j$$

$$\lambda_i(A - BK) = -2 \pm 2j$$

$$= \lambda_i\left(\begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right) = \lambda_i\left(\begin{bmatrix} 0 & 1 \\ \frac{-K_s - k_1}{M} & \frac{-K_d - k_2}{M} \end{bmatrix}\right)$$

$$= \left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ \frac{-K_s - k_1}{M} & \frac{-K_d - k_2}{M} \end{bmatrix}\right| = \left|\begin{bmatrix} s & -1 \\ \left(\frac{K_s + k_1}{M}\right) & \left(s + \frac{K_d + k_2}{M}\right) \end{bmatrix}\right|$$

$$= s\left(s + \frac{K_d + k_2}{M}\right) + \left(\frac{K_s + k_1}{M}\right) = s\left(s + \frac{K_d + k_2}{M}\right) + \left(\frac{K_s + k_1}{M}\right)$$

$$s = \frac{-\left(\frac{K_d + k_2}{M}\right) \pm \sqrt{\left(\frac{K_d + k_2}{M}\right)^2 - 4\left(\frac{K_s + k_1}{M}\right)}}{2}$$

$$-2 = \frac{-\frac{K_d + k_2}{M}}{2} \qquad k_2 = 4M - K_d$$

$$2 = \frac{\sqrt{\left(\frac{K_d + k_2}{M}\right)^2 - 4\left(\frac{K_s + k_1}{M}\right)}}{2}$$

$$16M = (K_d + k_2)^2 - 4(K_s + k_1)$$

$$4(K_s + k_1) = (K_d + 4M - K_d)^2 - 16M$$

$$k_1 = 4M^2 - 4M - K_s$$

$$K = \begin{bmatrix} 4M^2 - 4M - K_s & 4M - K_d \end{bmatrix}$$

To select the regulator values, component values must be selected.

$$M = 1kg \qquad K_s = 10\frac{N}{m} \qquad K_d = 1\frac{Ns}{m}$$

$$K = \begin{bmatrix} 4M^2 - 4M - K_s & 4M - K_d \end{bmatrix}$$

$$K = \begin{bmatrix} 4(1)^2 - 4(1) - 10 & 4(1) - (1) \end{bmatrix}$$

$$K = \begin{bmatrix} -10 & 3 \end{bmatrix}$$

The desired response time for the system will be a time response of approximately 0.2sec.

$$G(s) = \frac{x}{F} = \frac{1}{Ms^2 + K_d s + K_s}$$

$$A_{cl} = A - BK = \begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K_s - k_1}{M} & \frac{-K_d - k_2}{M} \end{bmatrix}$$

$$\Phi_e(s) = |sI - A_{cl}| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ \frac{-K_s - k_1}{M} & \frac{-K_d - k_2}{M} \end{bmatrix} \right| = \left| \begin{matrix} s & 1 \\ \frac{K_s + k_1}{M} & \frac{K_d + k_2}{M} + s \end{matrix} \right|$$

$$= s^2 + s\left(\frac{K_d + k_2}{M}\right) + \left(\frac{-K_s - k_1}{M}\right) = (s + 0.2)^2$$

$$\frac{-K_s - k_1}{M} = (0.2)^2 \qquad k_1 = -10.04$$

$$\frac{K_d + k_2}{M} = 2(0.2) \qquad k_2 = -0.6$$

The estimator may then be designed using Ackermann's formula.

$$L = \Phi_e(A)\begin{bmatrix} C \\ CA \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix} + 0.2\right)^2 \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix}^2 + 2(0.2)\begin{bmatrix} 0 & 1 \\ \frac{-K_s}{M} & \frac{-K_d}{M} \end{bmatrix} + 0.2^2 I\right)\begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \end{bmatrix} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \left(\begin{bmatrix} \frac{-K_s}{M} & \frac{-K_d}{M} \\ \frac{K_s K_d}{M^2} & \frac{-K_s}{M} + \frac{K_d^2}{M^2} \end{bmatrix} + \begin{bmatrix} 0 & 0.4 \\ \frac{-0.4K_s}{M} & \frac{-0.4K_d}{M} \end{bmatrix} + \begin{bmatrix} 0.04 & 0 \\ 0 & 0.04 \end{bmatrix}\right)\begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \end{bmatrix} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-K_s}{M} + 0.04 & \frac{-K_d}{M} + 0.4 \\ \frac{K_s K_d}{M^2} - \frac{0.4K_s}{M} & \frac{-K_s}{M} + \frac{K_d^2}{M^2} - \frac{0.4K_d}{M} + 0.04 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-K_s}{M} + 0.04 & \frac{-K_d}{M} + 0.4 \\ \frac{K_s K_d}{M^2} - \frac{0.4K_s}{M} & \frac{-K_s}{M} + \frac{K_d^2}{M^2} - \frac{0.4K_d}{M} + 0.04 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-K_d}{M} + 0.4 \\ \frac{-K_s}{M} + \frac{K_d^2}{M^2} - \frac{0.4K_d}{M} + 0.04 \end{bmatrix} = \begin{bmatrix} -0.6 \\ -9.36 \end{bmatrix}$$

The compensator coefficients may then be calculated.

$$A_c = A - BK - LC$$

$$A_c = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \begin{bmatrix} -10 & 3 \end{bmatrix} - \begin{bmatrix} -0.6 \\ -9.36 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$A_c = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -10 & 3 \end{bmatrix} - \begin{bmatrix} -0.6 & 0 \\ -9.36 & 0 \end{bmatrix} = \begin{bmatrix} 0.6 & 1 \\ 9.36 & -4 \end{bmatrix}$$

$$B_c = L = \begin{bmatrix} -0.6 \\ -9.36 \end{bmatrix}$$

$$C_c = K = \begin{bmatrix} -10 & 3 \end{bmatrix}$$

The compensator transfer function can then be written,

$$G_c(s) = C_c(sI - A_c)^{-1} B_c$$

$$= \begin{bmatrix} -10 & 3 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0.6 & 1 \\ 9.36 & -4 \end{bmatrix} \right)^{-1} \begin{bmatrix} -0.6 \\ -9.36 \end{bmatrix} = \begin{bmatrix} -10 & 3 \end{bmatrix} \begin{bmatrix} s - 0.6 & -1 \\ -9.36 & s - 4 \end{bmatrix}^{-1} \begin{bmatrix} -0.6 \\ -9.36 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} -10 & 3 \end{bmatrix} \left\| \begin{matrix} s - 0.6 & -1 \\ -9.36 & s - 4 \end{matrix} \right\| \begin{bmatrix} -0.6 \\ -9.36 \end{bmatrix}}{\begin{vmatrix} s - 0.6 & -1 \\ -9.36 & s - 4 \end{vmatrix}} = \frac{\begin{bmatrix} -10 & 3 \end{bmatrix} \begin{bmatrix} s - 4 & 9.36 \\ 1 & s - 0.6 \end{bmatrix}^T \begin{bmatrix} -0.6 \\ -9.36 \end{bmatrix}}{(s - 0.6)(s - 4) - (-1)(-9.36)}$$

$$= \frac{\begin{bmatrix} -10 & 3 \end{bmatrix} \begin{bmatrix} s - 4 & 1 \\ 9.36 & s - 0.6 \end{bmatrix} \begin{bmatrix} -0.6 \\ -9.36 \end{bmatrix}}{1s^2 - 4.6s - 6.96} = \frac{\begin{bmatrix} -10 & 3 \end{bmatrix} \begin{bmatrix} -0.6s - 6.96 \\ -9.36s \end{bmatrix}}{1s^2 - 4.6s - 6.96}$$

$$= \frac{-22.08s + 69.6}{s^2 - 4.6s - 6.96}$$

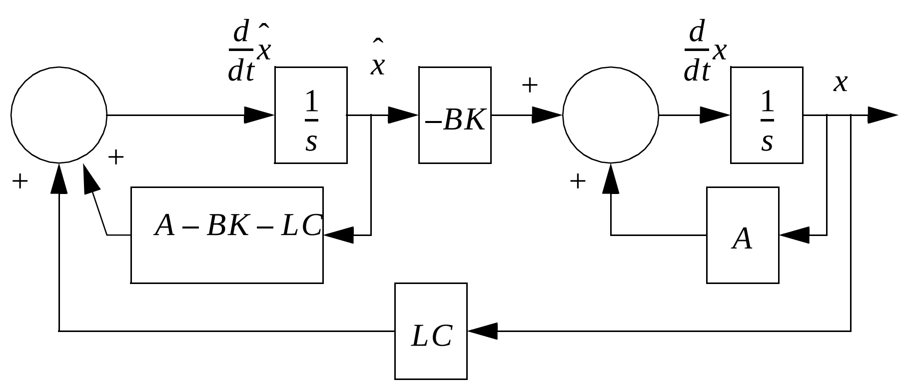
- This system can be implemented with a C program as shown below,

```
#define     PERIOD         0.001   // period of the interrupt
interrupt_loop(){ // this routine is called at a regular interval PERIOD

}

double analog_in(){

}
```



21.6 LQR

- Linear Quadratic Regulator (LQR) allows the design of a regulating system that tends to zero, but also considers the control effort.

- The basic function for evaluating the controller is defined below. The integral is evaluated to reduce the value. Ideally the state value will go to zero (it is a regulator), and the plant input will be minimized. It is assumed that D=0. Here a new state 'z' is defined. It should reflect the system state values that should be regulated to zero.

$$J_{LQR} = \int_0^{\infty} [(y(t))^T y(t) + r(u(t))^2] dt$$

$$(y(t))^T y(t) = \text{state cost}$$

$$(u(t))^2 = \text{control cost}$$

$$r = \text{control penalty}$$

$$z = C_z x$$

$$J_{LQR} = \int_0^{\infty} [(x(t))^T C_z^T C_z x(t) + r(u(t))^2] dt$$

- the design process involves adjusting the r value to obtain the desired result.

-

Assume a simple controller form,

$$u(t) = -K_{LQR}x(t)$$

Solve using an Algebraic Riccati Equation

Consider a simple system

$$\dot{x} = Ax + Bu \qquad\qquad y = Cx$$

The equivalent transfer function can be written, where a(s) is the characteristic eqn.

$$C(sI - A)^{-1}B = \frac{b(s)}{a(s)} = \frac{b(s)}{|sI - A|}$$

This is followed by.......

UNKNOWN STUFF

The desired charactersitic equation can be used to find the controller gain values.

$$|sI - A + BK_{LQR}| = \prod_{i=1}^{n}(s - p_i)$$

$$\Delta(s) = a(s)a(-s) + r^{-1}b(s)b(-s)$$

The poles are then found using Ackermann's method.

$$(u(t))^2 = \text{control cost}$$

$$r = \text{control penalty}$$

$$J_{LQR} = \int_0^{\infty}[(x(t))^T C^T Cx(t) + r(u(t))^2]dt$$

$$J_{LQR} = \int_0^{\infty}[(x(t))^T C^T Cx(t) + r(u(t))^2]dt$$

21.7 LINEAR QUADRATIC GAUSSIAN (LQG) COMPENSATORS

• Uses an optimal regulator and estimator.

• Stability is guaranteed if the model is accurate. Verification is needed

• The design focuses on,

A cost function for the system state, the design tries to minimize this
The relative weighting for the process and sensor noise

•

21.8 VERIFYING CONTROL SYSTEM STABILITY

• Introduce random errors to the system model and test for stability

• Use the Nyquist Stability Theorem,

$Z = N + P$

where,

P = poles of G(s)Gc(s) in the right hand plane

Z = closed loop poles in the right hand plane

N = number of clockwise loops in the Nyquist Diagram about -1

$Z = 0, N = -P$ For stability

Note: Larger loops indicate a more stable system. Tight loops are very sensitive to parameter variations.

21.8.1 Stability

- The stability function is defined using the Nyquist plot.

Define the points for the Nyquist plot

$$L(s) = G(s)G_c(s)$$

$$L_N(s) = G_N(s)G_c(s)$$

$$L_A(s) = G_A(s)G_c(s)$$

The distance, 'd', from the point at -1 to the calculated point is found as a function.

$$d(j\omega) = L_N(j\omega) - (-1 + 0j)$$

The sensistivity 'S' is defined as the inverse of the distance from the point at -1.

$$S(j\omega) = \frac{1}{d(j\omega)} = \frac{1}{1 + L_N(j\omega)}$$

21.8.2 Bounded Gain

• In practical terms system gain is limited by real components. As a result it is important to ensure that the design will not exceed the maximum values available.

The boundedness of the system can be tested by ensuring that the sensitivity always remains less than some given value 'gamma' for all frequencies. Note: this requires that D must be smaller than gamma.

$$|S(j\omega)| < \gamma$$

$$C(j\omega I - A)^{-1}B + D < \gamma$$

The Hamiltonian matrix is defined below. It does not have purely complex Eigenvalues.

$$H = \begin{bmatrix} A + B(\gamma^2 I - D^T D)^{-1} D^T C & B(\gamma^2 I - D^T D)^{-1} B^T \\ -C^T(I + D(\gamma^2 I - D^T D)^{-1} D^T)C & -A^T - C^T D(\gamma^2 I - D^T D)^{-1} B^T \end{bmatrix}$$

Or, if D=0,

$$H = \begin{bmatrix} A & \frac{BB^T}{\gamma^2} \\ -C^T C & -A^T \end{bmatrix}$$

This matrix can then be tested, if there are no purely complex eigenvalues, the system is bounded.

$$|\lambda I - H| = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} A & \frac{BB^T}{\gamma^2} \\ -C^T C & -A^T \end{bmatrix} \right| = \left| \begin{bmatrix} \lambda - A & -\frac{BB^T}{\gamma^2} \\ C^T C & \lambda + A^T \end{bmatrix} \right|$$

$$= \lambda^2(1) + \lambda(-A + A^T) + \left(-\frac{BB^T C^T C}{\gamma^2}\right)$$

etc....

The algebraic Ricatti equation can also be used

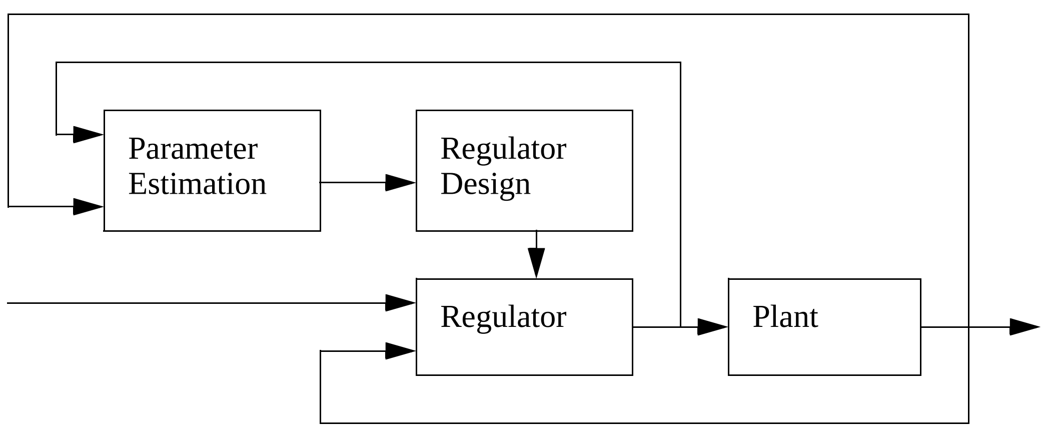
$$A^T X + XA + C^T C + \frac{XBB^T X}{\gamma^2} = 0$$

21.9 ADAPTIVE CONTROLLERS

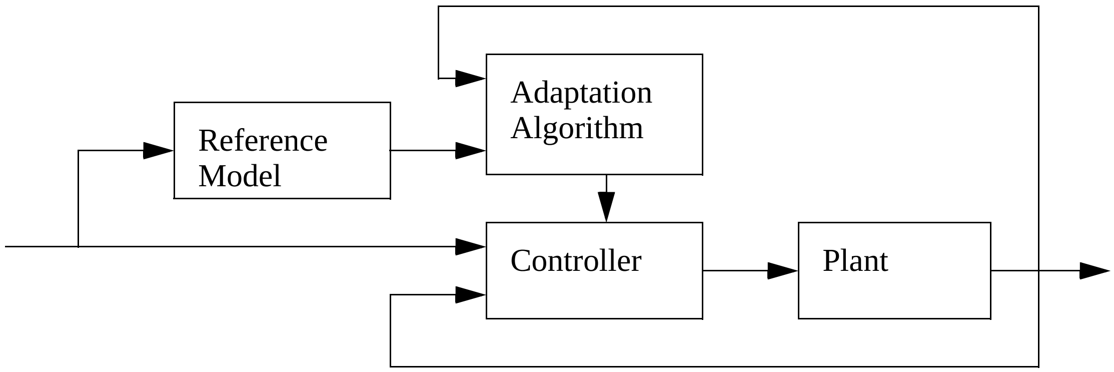
• There are two basic types of adaptive control,

Self Tuining Regulators
Model Reference Adaptive Control (MRAC)

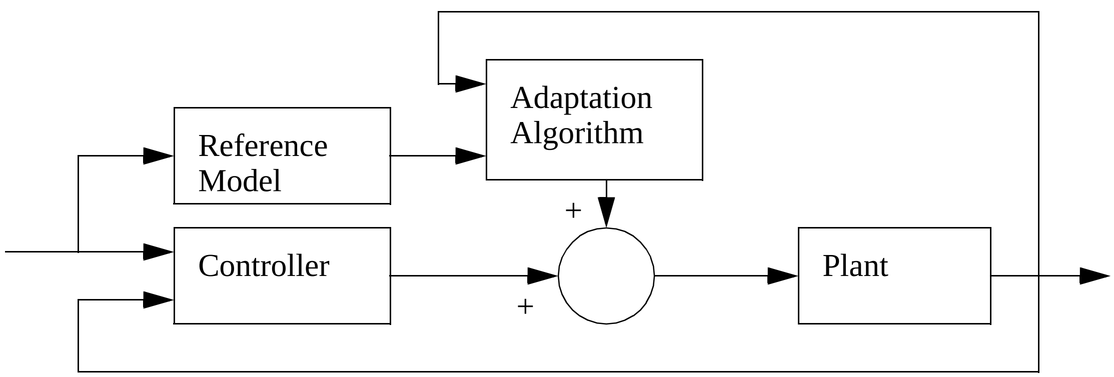
• A self tuning regulator



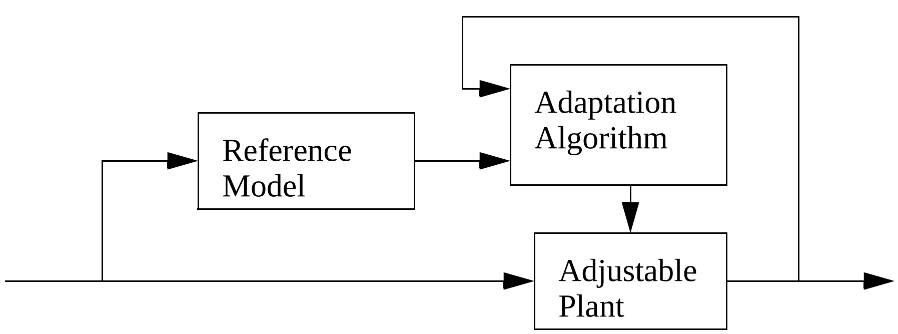
• Parameter Adaptive MRAC



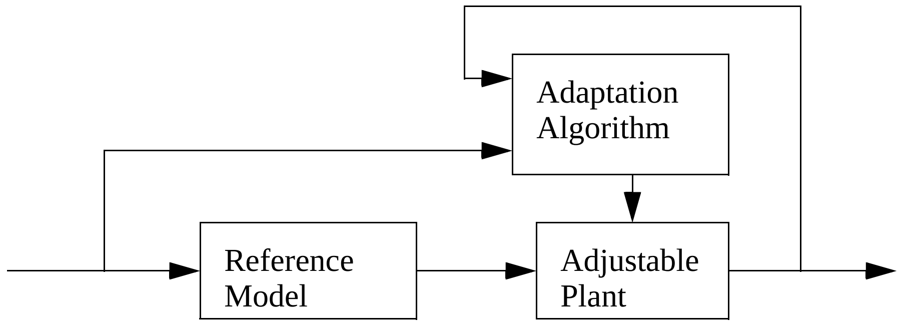
• Signal Synthesis MRAC



• Parallel MRAC



• Series MRAC



• The basic MRAC system is defined below,

The reference model is developed from the fundamentals of the system. It should be controllable and stable.

$$\frac{d}{dt}X_m = A_m X_m + B_m r$$

Parameters can be adpated using,

$$e = X_m - X_p$$
$$\frac{d}{dt}X_p = A_p X_p + B_p r$$

Where,

$$A_p, B_p = \text{Adjustable parameter matrices}$$

$$A_m, B_m = \text{Reference model matrices}$$

$$\frac{d}{dt}X_p = A_p X_p + B_p(r + u_a)$$

$$\frac{d}{dt}e = A_m X_m + B_m r - A_p X_p - B_p(r + u_a) = A_m(e + X_p) + B_m r - A_p X_p - B_p(r + u_a)$$

$$\frac{d}{dt}e = e(A_m) + X_p(A_m - A_p) + r(B_m - B_p) + u_a(-B_p)$$

$$\frac{d}{dt}e = e(A_m) + f \qquad f = X_p(A_m - A_p) + r(B_m - B_p) + u_a(-B_p)$$

A Lyapunov can be defined for the error

$$V = e^T Q e$$

$$\frac{d}{dt}V = \left(\frac{d}{dt}e^T\right)Qe + e^T Q\left(\frac{d}{dt}e\right) = (e^T(A_m^T) + f^T)Qe + e^T Q(e(A_m) + f)$$

$$\frac{d}{dt}V = e^T(A_m^T Q + QA_m)e + 2e^T Qf$$

$$\frac{d}{dt}V = -e^T We + 2e^T Qf \qquad W = -(A_m^T Q + QA_m)$$

21.10 OTHER METHODS

H-infinity and mu methods are designed to shape the sensitivity plot

21.10.1 Kalman Filtering

• Kalman Filter helps settle estimators using estimates of snesor and process noise.

$$\frac{d}{dt}x = Ax + Bu + B_w w$$

$$y = Cx + v$$

where,

B_w = process noise weighting matrix

w = process noise matrix

v = sensor noise

$$G_{yw}(s) = C(sI - A)^{-1}B_w = \frac{N(s)}{D(s)}$$

$$D(s)D(-s) \pm \frac{R_w}{R_v}N(s)N(-s) = 0$$

where,

R_w, R_v = factors related to process/sensors noise

21.11 SUMMARY

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21.12 PRACTICE PROBLEMS

21.13 PRACTICE PROBLEM SOLUTIONS

21.14 ASSIGNMENT PROBLEMS

1.