20. STATE SPACE ANALYSIS

Topics:

Objectives:

20.1 INTRODUCTION

- state equations can be converted to transfer functions. The derivation follows.

State equations as functions of time

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

In the s-domain

$$sX - X_0 = AX + BU$$

$$X(sI - A) = BU + X_0$$

$$X = (sI - A)^{-1}BU + (sI - A)^{-1}X_0$$

$$Y = CX + DU$$

$$Y = C((sI - A)^{-1}BU + (sI - A)^{-1}X_0) + DU$$

$$Y = (C(sI - A)^{-1}B + D)U + C(sI - A)^{-1}X_0$$

Assuming the system starts at rest,

$$Y = (C(sI - A)^{-1}B + D)U$$

$$\frac{Y}{U} = (C(sI - A)^{-1}B + D) \qquad \text{(the transfer function)}$$

- state equation coefficient matrices can be transformed to another equivalent for, if the state vector is rearranged.

State equations as functions of time

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Map one state vector to another one

$$x = Tz$$

This can be used to calculate new coefficient matrices

$$\dot{z} = T^{-1}(Ax + Bu)$$

$$\dot{z} = T^{-1}(ATz + Bu)$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu \qquad \bar{A} = T^{-1}AT \qquad \bar{B} = T^{-1}B$$

$$\dot{z} = \bar{A}z + \bar{B}u$$

The output equation becomes

$$y = Cx + Du$$

$$y = CTz + Du$$

$$y = CTz + Du \qquad \bar{C} = CT \qquad \bar{D} = D$$

$$y = \bar{C}z + \bar{D}u$$

The equivalent set of state equations is,

$$\dot{z} = \bar{A}z + \bar{B}u \qquad \bar{A} = T^{-1}AT \qquad \bar{B} = T^{-1}B$$

$$y = \bar{C}z + \bar{D}u \qquad \bar{C} = CT \qquad \bar{D} = D$$

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The two following transfer functions are equivalents

$$G = C(sI-A)^{-1}B + D$$

$$\bar{G} = \bar{C}(sI-\bar{A})^{-1}\bar{B} + \bar{D}$$

This can be shown by applying the transformation matrix

$$G = CTT^{-1}(sI-A)^{-1}TT^{-1}B + TT^{-1}D$$

$$G = (CT)(T^{-1}(sI-A)^{-1}T)(T^{-1}B) + D$$

$$G = (CT)((sI-A)^{-1})(T^{-1}B) + D$$

$$G = \bar{C}((sI-\bar{A})^{-1})\bar{B} + \bar{D} = \bar{G}$$

• The transfer function form can be put into a matrix form. In this case the denominator is the characteristic equation.

The transfer function can be said to be equivalant to the determinants of the matrix form.

$$\frac{\begin{vmatrix} (sI-A) & -B \\ C & D \end{vmatrix}}{|sI-A|} = \frac{(sI-A)D-(-B)C}{sI-A} = (sI-A)D + BC(sI-A)$$

$$G = C(sI-A)^{-1}B + D = \frac{\begin{vmatrix} (sI-A) & -B \\ C & D \end{vmatrix}}{|sI-A|} = \frac{poles}{zeros}$$

$$|sI-A| = \text{characteristic equation} = \text{homogeneous}$$

• The free (homogeneous) response of a system can be used to find the state transition matrix.

The homogeneous equation can be written in the s-domain, and then converted to time.

$$X_h = |sI - A|X_0$$

$$x_h(t) = L^{-1}[|sI - A|X_0]$$

$$x_h(t) = L^{-1}\left[\left[\frac{I}{s} + \frac{A}{s} + \frac{A^2}{s^2} + \frac{A^3}{s^3} + \dots\right]X_0\right]$$

$$x_h(t) = e^{At}x_0$$

$$\boxed{e^{At} = \text{transition matrix}}$$

aside: This expansion is a McLaurin (Taylor) series.

$$e^{At} = I + At + \left(\frac{1}{2!}\right)A^2t^2 + \left(\frac{1}{3!}\right)A^3t^3 + \dots$$

• The forced response (particular) response of the system can be found using convolution,

The homogeneous equation can be written in the s-domain, and then converted to time.

$$\dot{x} = Ax + Bu$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y = Cx + Du$$

$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{initial response}} + \int_0^t \underbrace{Ce^{A(t-\tau)}}_{\text{impulse response}}Bu(\tau)d\tau + Du(t)$$

• As an example the homogeneous/free response of the system is shown below.

Given a second order differential equation, the state matrix can be found,

$$F = M\ddot{x}$$

$$\dot{x} = v \qquad \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}F \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\dot{v} = \frac{F}{M}$$

This can be used to find the inverse matrix,

$$(sI - A)^{-1} = \left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}}{s^2 - 0} = \begin{bmatrix} \frac{s}{s^2} & \frac{1}{s^2} \\ \frac{0}{s^2} & \frac{s}{s^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

The function of time can be found assuming an initial position of 10 and velocity of 5.

$$e^{At} = L^{-1}[(sI - A)^{-1}] = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$x_h(t) = \begin{bmatrix} x \\ v \end{bmatrix} = e^{At}\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 + 5t \\ 5 \end{bmatrix}$$

• The forced/particular solution is shown below,

Given a second order differential equation, the state matrix can be found,

$$F = M\ddot{x}$$

• If a matrix is diagonizable, the diagonal matrix can be found with the following technique. This can be used for more advanced analysis techniques to create diagonal (and separable) system matrices.

Given a matrix A, a purely diagonal matrix may be found,

$$T^{-1}AT = \Lambda \qquad A = T\Lambda T^{-1} \qquad \Lambda = \begin{bmatrix} \lambda_1 & \dots & \\ \dots & \dots & \\ & & \lambda_n \end{bmatrix}$$

This may also be written as,

$$e^{At} = Te^{\Lambda t}T^{-1} \qquad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & \dots & \\ \dots & \dots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

This can also be applied to the

$$e^{At} = I + At + \frac{1}{2!}A^2 t^2 + \dots$$

$$e^{At} = I + T\Lambda T^{-1}t + \frac{1}{2!}(T\Lambda T^{-1})^2 t^2 + \dots$$

Eigenvalues can be found using the following relationship. If any of the Eigenvalues are repeated the Jordan normal form is required.

$$|\lambda I - A| = 0$$

> Note: the Eigenvalues are the values found in the characteristic/homogeneous equation solution.

The Eigenvalues are then used to find the Eigenvectors,

$$(\lambda I - A)v = 0 \qquad Av = \lambda v$$

The Eigenvectors are then combined into a single matrix, this is the transition matrix.

- example,

Given the state matrix

$$A = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$$

The eigenvalues can be calculated,

$$|\lambda I - A| = \left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \right| = \left| \begin{bmatrix} \lambda + 1 & -2 \\ 3 & \lambda - 4 \end{bmatrix} \right|$$

$$\therefore \quad = (\lambda + 1)(\lambda - 4) - (-6) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

Therefore the Eigenvalues are, $\lambda_1 = 2 \qquad \lambda_2 = 1$

An eigenvector can be calculated using the Eigenvalue 2.

$$\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \lambda_1 \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = 2 \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}$$

$$-u_{11} + 2u_{21} = 2u_{11} \qquad \therefore 2u_{21} = 3u_{11} \qquad v_1 = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Another eigenvector can be calculated using the Eigenvalue 1.

$$\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \lambda_2 \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = 1 \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix}$$

$$-u_{12} + 2u_{22} = u_{12} \qquad \therefore u_{22} = u_{12} \qquad v_2 = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The Eigenvectors can be combined to a single matrix.

$$T = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

To check, $Av_i = \lambda_i v_i$

$$A \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}, A \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}, 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

verified

- If there are repeated Eigenvalues in the system the Jordan Form can be used.

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_{n-1} & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix} \qquad \text{Jordan block}$$

Consider the example below with repeated eigenvalues.

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \qquad \text{Find a matrix, that has repeated eigenvalues, and is not singular}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & -1 \\ -1 & \lambda - 2 & -1 \\ -1 & -1 & \lambda - 2 \end{vmatrix} = (\lambda + 3)^2(\lambda - 0)$$

Solve for the Jordan block one column at a time,

$$1\begin{bmatrix} T_{11} \\ T_{21} \\ T_{31} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} T_{11} \\ T_{21} \\ T_{31} \end{bmatrix} = \begin{bmatrix} -2T_{11} + T_{21} + T_{31} \\ T_{11} - 2T_{21} + T_{31} \\ T_{11} + T_{21} - 2T_{31} \end{bmatrix}$$

FIX

$$T_{11} = 2T_{11} + T_{21} + T_{31} \qquad T_{21} =$$

$$T_{21} = T_{11} + 2T_{21} + T_{31} \qquad T_{21} =$$

$$T_{31} = T_{11} + T_{21} + 2T_{31} \qquad T_{31} =$$

$$1\begin{bmatrix} T_{12} \\ T_{22} \\ T_{32} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} T_{12} \\ T_{22} \\ T_{32} \end{bmatrix} = \begin{bmatrix} -2T_{12} + T_{22} + T_{32} \\ T_{12} - 2T_{22} + T_{32} \\ T_{12} + T_{22} - 2T_{32} \end{bmatrix}$$

$$T_{12} = 2T_{12} + T_{22} + T_{32}$$

$$T_{22} = T_{12} + 2T_{22} + T_{32}$$

$$T_{32} = T_{12} + T_{22} + 2T_{32}$$

$$0\begin{bmatrix} T_{13} \\ T_{23} \\ T_{33} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} T_{13} \\ T_{23} \\ T_{33} \end{bmatrix} = \begin{bmatrix} -2T_{13} + T_{23} + T_{33} \\ T_{13} - 2T_{23} + T_{33} \\ T_{13} + T_{23} - 2T_{33} \end{bmatrix}$$

$$4T_{13} = 2T_{13} + T_{23} + T_{33}$$

$$4T_{23} = T_{13} + 2T_{23} + T_{33}$$

$$4T_{33} = T_{13} + T_{23} + 2T_{33}$$

• The Eigenvectors can be used to calculate the system response.

Given the following Eigen values and vectors the free response is,

$$\lambda_1 = 2 \qquad \lambda_2 = 1 \qquad v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_h(t) = \sum_{i=1}^{n} v_i e^{\lambda_i t} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{1t} = \begin{bmatrix} 2e^{2t} + e^{t} \\ 3e^{2t} + e^{t} \end{bmatrix}$$

• zeros of state space functions can be found using the state matrices.

The following relationship locates the zero frequencies in state equations.

$$\begin{vmatrix} s_0 I - A & -B \\ C & D \end{vmatrix} = 0$$

Consider the example beginning with state equations,

$$\dot{x} = v - 5F$$

$$\dot{v} = -3x - 4v + 5F$$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} -5 \\ 5 \end{bmatrix}F \qquad x = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + 0F$$

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} -5 \\ 5 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$

This can be put into the matrix form,

$$\begin{vmatrix} \left(s_0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\right) & -\begin{bmatrix} -5 \\ 5 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & 0 \end{vmatrix} = 0$$

$$\begin{bmatrix} s_0 & -1 & 5 \\ 3 & s_0 + 4 & -5 \\ 1 & 0 & 0 \end{bmatrix} = 0$$

$$s_0(0) - (-1)(-4.5) + 5(-(s_0 + 4)) = 0$$

$$-24.5 = 5s_0$$

$$s_0 = -4.9$$

There is one zero at -4.9 rad/sec. The order of the polynomial determines the number of zeros. If the polynomial has no ’s’ terms, then there are no zeros.

20.2 OBSERVABILITY

- a system is observable iff the system state x(t) can be found by observing the input u and output y over a period of time from x(t) to x(t+h).

- If an input is to be observable it must be detectable in the output. For example consider the following state equations.

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} u$$

$$y = \begin{bmatrix} g & h \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} j \\ k \end{bmatrix} u$$

if the parameters c, d or f were not in the final expression for y, the second state equation would be said to be not observable. This will happen when an output parameter is zero in the C matrix and an equation is decoupled in the A matrix, i.e. the same variable column is zero.

For example,

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -a \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 3 \\ b \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 6 \end{bmatrix} u$$

$$G = C(sI-A)^{-1}B + D = \begin{bmatrix} 5 & 0 \end{bmatrix}\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -a \end{bmatrix}\right)^{-1}\begin{bmatrix} 3 \\ b \end{bmatrix} + \begin{bmatrix} 6 \end{bmatrix}$$

$$G = \begin{bmatrix} 5 & 0 \end{bmatrix}\left(\begin{bmatrix} s-1 & 0 \\ 0 & s+a \end{bmatrix}\right)^{-1}\begin{bmatrix} 3 \\ b \end{bmatrix} + \begin{bmatrix} 6 \end{bmatrix} = \begin{bmatrix} 5 & 0 \end{bmatrix}\left(\frac{\begin{bmatrix} s+a & 0 \\ 0 & s-1 \end{bmatrix}}{(s-1)(s+a)}\right)\begin{bmatrix} 3 \\ b \end{bmatrix} + \begin{bmatrix} 6 \end{bmatrix}$$

$$G = \begin{bmatrix} 5 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s+a} \end{bmatrix}\begin{bmatrix} 3 \\ b \end{bmatrix} + \begin{bmatrix} 6 \end{bmatrix} = \begin{bmatrix} 5 & 0 \end{bmatrix}\begin{bmatrix} \frac{3}{s-1} \\ \frac{b}{s+a} \end{bmatrix} + \begin{bmatrix} 6 \end{bmatrix} = \frac{15}{s-1} + 6$$

Note: the variables ’a’ and ’b’ both dissapeared because this system is not observable.

• This often happens when a system has elements that are decoupled, or when a pole and zero cancel each other.

• Observability can be verified formally for an LTI system with the following relationship.

The system is unobservable if the equation below is true for any state,

$$Ce^{At}x^* = 0 \qquad t \geq 0 \qquad x^* = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \dots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} 0 \\ 0 \\ \dots \\ 1 \end{bmatrix}$$

One example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 5 & 6 \end{bmatrix}$$

$$Ce^{At}x^* = \begin{bmatrix} 5 & 6 \end{bmatrix} \begin{bmatrix} e^t & e^{2t} \\ e^{3t} & e^{4t} \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} (5e^t + 6e^{3t}) & (5e^{2t} + 6e^{4t}) \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$Ce^{At}x^* = \left\{ \begin{bmatrix} (5e^t + 6e^{3t}) & (5e^{2t} + 6e^{4t}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} (5e^t + 6e^{3t}) & (5e^{2t} + 6e^{4t}) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$Ce^{At}x^* = \{ (5e^t + 6e^{3t}), (5e^{2t} + 6e^{4t}) \}$$

Both values are non-zero, so both states are observable.

One example,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 0 \end{bmatrix}$$

$$Ce^{At}x^* = \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} 3e^t & 0 \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \{ 3e^t, 0 \}$$

The second state is not observable.

• Another theorem for testing observability is given below. If any of the states satisfies the equation it is unobservable.

$$\begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix} x^* = 0$$

• Yet another test for observability is,

$$rank(M_o) = rank\begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix} = n$$

• If a system in unobservable, it is possible to make it observable by changing the model.

• A pole-zero cancellation is often the cause of the loss of observability.

• If all unstable modes are observable, the system is detectable.

20.3 CONTROLLABILITY

- a system is controllable iff there is an input u(t) that will cause the system to go from any initial state to any final state in a finite time.

- stabilizable if it is controllable or if the uncontrollable nodes are stable.

- If an input is to be observable it must be detectable in the output. For example consider the following state equations.

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}u$$

$$y = \begin{bmatrix} g & h \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} j \\ k \end{bmatrix}u$$

if the parameters c, d or f were not in the final expression for y, the second state equation would be said to be not uncontrollable. This will happen when an output parameter is zero in the C matrix and an equation is decoupled in the A matrix, i.e. the same variable column is zero.

For example,

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & -b \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}u$$

$$y = \begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 5 \end{bmatrix}u$$

$$G = C(sI - A)^{-1}B + D = \begin{bmatrix} 3 & 4 \end{bmatrix}\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ a & -b \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \end{bmatrix}$$

$$G = \begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} s-1 & 0 \\ 0 & s+b \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \end{bmatrix}\frac{\begin{bmatrix} s+B & 0 \\ 0 & s-1 \end{bmatrix}}{(s-1)(s+b)}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \end{bmatrix}$$

$$G = \begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s+b} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{s-1} & \frac{4}{s+b} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \end{bmatrix} = \frac{6}{s-1} + 5$$

Note: the variables ’a’ and ’b’ both dissapeared because this system is not controllable.

• This can be verified with the

The basic relationship is,

$$(x^*)^T e^{At} B = 0 \qquad\qquad t \geq 0$$

An example is,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}^T e^{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} t} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} e^{1t} & e^{2t} \\ e^{3t} & e^{4t} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} 5e^{1t} + 6e^{2t} \\ 5e^{3t} + 6e^{4t} \end{bmatrix} = \{ 5e^{1t} + 6e^{2t}, 5e^{3t} + 6e^{4t} \}$$

The results are both non-zero, so both states are controllable.

An example is,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}^T e^{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} t} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} e^{1t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} 0 \\ 3e^{2t} \end{bmatrix} = \{ 0, 3e^{2t} \}$$

The first state has a zero, so it in not controllable.

• Another test for controllability is,

$$(x^*)^T \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}$$

• Yet another test for controllability is,

$$rank(M_c) = rank\begin{bmatrix} B \ AB \ \dots \ A^{n-1}B \end{bmatrix} = n$$

• For a system to be controllable, all of the states must be controllable.

• If a system in uncontrollable, it is possible to make it observable by changing the model.

• A pole-zero cancellation is often the cause of the loss of observability.

• if all unstable modes are controllable, the systems is said to be stabilizable.

• The principle of Duality requires that a system be completely observable to be controllable.

20.4 OBSERVERS

• Observers estimate system state variables when not all of the variables are directly observable.

• Observers use a limited set of system states that are available to identify other system states that are not observable.

• The separation principle ensures that the observer cannot effect the stability of the system it is observing.

20.5 SUMMARY

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20.6 PRACTICE PROBLEMS

20.7 PRACTICE PROBLEM SOLUTIONS

20.8 ASSIGNMENT PROBLEMS

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20.9 BIBLIOGRAPHY

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