5. ROTATION

Topics:

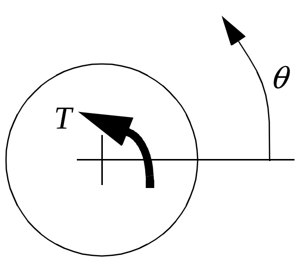
- Basic laws of motion
- Inertia, springs, dampers, levers, gears and belts
- Design cases

Objectives:

- To be able to develop and analyze differential equations for rotational systems.

5.1 INTRODUCTION

The equations of motion for a rotating mass are shown in Figure 5.1. Given the angular position, the angular velocity can be found by differentiating once, the angular acceleration can be found by differentiating again. The angular acceleration can be integrated to find the angular velocity, the angular velocity can be integrated to find the angular position. The angular acceleration is proportional to an applied torque, but inversely proportional to the mass moment of inertia.



equations of motion

$$\omega = \left(\frac{d}{dt}\right)\theta \tag{1}$$

$$\alpha = \left(\frac{d}{dt}\right)\omega = \left(\frac{d}{dt}\right)^2\theta \tag{2}$$

OR

$$\theta(t) = \int \omega(t)dt = \int\int \alpha(t)dtdt \tag{3}$$

$$\omega(t) = \int \alpha(t)dt \tag{4}$$

$$\alpha(t) = \frac{T(t)}{J_M} \tag{5}$$

where,

θ, ω, α = *position, velocity and acceleration*
J_M = *second mass moment of inertia of the body*
T = *torque applied to body*

Figure 5.1 Basic properties of rotation

Note: A 'torque' and 'moment' are equivalent in terms of calculations. The main difference is that 'torque' normally refers to a rotating moment.

Given the initial state of a rotating mass, find the state 5 seconds later.

$$\theta_0 = 1rad \qquad \omega_0 = 2\frac{rad}{s} \qquad \alpha = 3\frac{rad}{s^2}$$

ans. $\theta(5) = 86rad$
$\omega(5) = 17\frac{rad}{s}$

Figure 5.2 Drill problem: Find the position with the given conditions

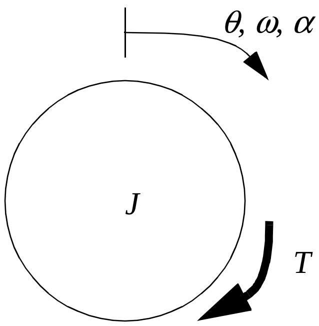
5.2 MODELING

Free Body Diagrams (FBDs) are required when analyzing rotational systems, as they were for translating systems. The force components normally considered in a rotational system include,

- inertia - opposes acceleration and deceleration
- springs - resist deflection
- dampers - oppose velocity
- levers - rotate small angles
- gears and belts - change rotational speeds and torques

5.2.1 Inertia

When unbalanced torques are applied to a mass it will begin to accelerate, in rotation. The sum of applied torques is equal to the inertia forces shown in Figure 5.3.



$$\sum T = J_M \alpha \qquad (6)$$

$$J_M = I_{xx} + I_{yy} \qquad (7)$$

$$I_{xx} = \int y^2 dM \qquad (8)$$

$$I_{yy} = \int x^2 dM \qquad (9)$$

Note: The 'mass' moment of inertia will be used when dealing with acceleration of a mass. Later we will use the 'area' moment of inertia for torsional springs.

Figure 5.3 Summing moments and angular inertia

The mass moment of inertia determines the resistance to acceleration. This can be calculated using integration, or found in tables. When dealing with rotational acceleration it is important to use the mass moment of inertia, not the area moment of inertia.

The center of rotation for free body rotation will be the centroid. Moment of inertia values are typically calculated about the centroid. If the object is constrained to rotate about some point, other than the centroid, the moment of inertia value must be recalculated. The parallel axis theorem provides the method to shift a moment of inertia from a centroid to an arbitrary center of rotation, as shown in Figure 5.4.

$$J_M = \tilde{J}_M + Mr^2$$

where,

J_M = *mass moment about the new point*

$\tilde{J}_M$ = *mass moment about the center of mass*

M = *mass of the object*

r = *distance from the centroid to the new point*

Figure 5.4 Parallel axis theorem for shifting a mass moment of inertia

$$J_A = \tilde{J}_A + Ar^2$$

where,

J_A = *area moment about the new point*

$\tilde{J}_A$ = *area moment about the centroid*

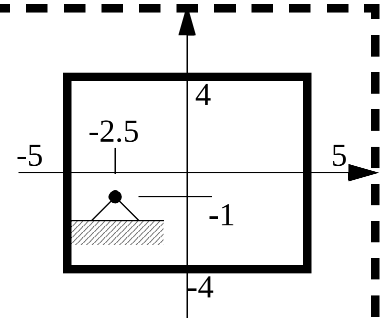
A = *mass of the object*

r = *distance from the centroid to the new point*

Figure 5.5 Parallel axis theorem for shifting a area moment of inertia

Aside: If forces do not pass through the center of an object, it will rotate. If the object is made of a homogeneous material, the area and volume centroids can be used as the center. If the object is made of different materials then the center of mass should be used for the center. If the gravity varies over the length of the (very long) object then the center of gravity should be used.

An example of calculating a mass moment of inertia is shown in Figure 5.6. In this problem the density of the material is calculated for use in the integrals. The integrals are then developed using slices for the integration element dM. The integrals for the moments about the x and y axes, are then added to give the polar moment of inertia. This is then shifted from the centroid to the new axis using the parallel axis theorem.



The rectangular shape to the right is constrained to rotate about point A. The total mass of the object is 10kg. The given dimensions are in meters. Find the mass moment of inertia.

First find the density and calculate the moments of inertia about the centroid.

$$\rho = \frac{10Kg}{2(5m)2(4m)} = 0.125Kgm^{-2}$$

$$I_{xx} = \int_{-4}^{4} y^2 dM = \int_{-4}^{4} y^2 \rho 2(5m) dy = 1.25Kgm^{-1} \left.\frac{y^3}{3}\right|_{-4}^{4}$$

$$\therefore = 1.25Kgm^{-1}\left(\frac{(4m)^3}{3} - \frac{(-4m)^3}{3}\right) = 53.33Kgm^2$$

$$I_{yy} = \int_{-5}^{5} x^2 dM = \int_{-5}^{5} x^2 \rho 2(4m) dx = 1Kgm^{-1} \left.\frac{x^3}{3}\right|_{-5}^{5}$$

$$\therefore = 1Kgm^{-1}\left(\frac{(5m)^3}{3} - \frac{(-5m)^3}{3}\right) = 83.33Kgm^2$$

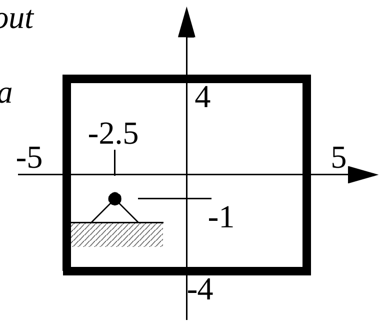
$$J_M = I_{xx} + I_{yy} = 53.33Kgm^2 + 83.33Kgm^2 = 136.67Kgm^2$$

The centroid can now be shifted to the center of rotation using the parallel axis theorem.

$$J_M = \tilde{J}_M + Mr^2 = 136.67Kgm^2 + (10Kg)((-2.5m)^2 + (-1m)^2) = 209.2Kgm^2$$

Figure 5.6 Mass moment of inertia example

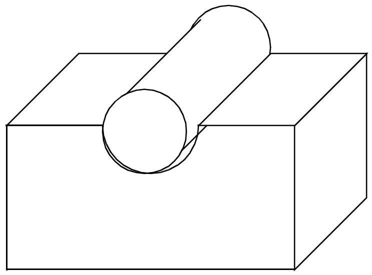
The rectangular shape to the right is constrained to rotate about point A. The total mass of the object is 10kg. The given dimensions are in meters. Find the mass moment of inertia WITHOUT using the parallel axis theorem.



ans.

$$I_{M_x} = 66.33Kgm^2$$
$$I_{M_y} = 145.8Kgm^2$$
$$J_M = 209.2Kgm^2$$

Figure 5.7 Drill problem: Mass moment of inertia calculation



The 20cm diameter 10 kg cylinder to the left is sitting in a depression that is effectively frictionless. If a torque of 10 Nm is applied for 5 seconds, what will the angular velocity be?

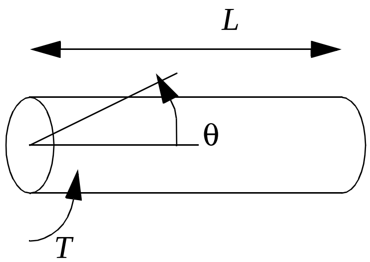
ans. $\theta(5s) = 312.5rad$

$\omega(5s) = 125\frac{rad}{s}$

Figure 5.8 Drill problem: Find the velocity of the rotating shaft

5.2.2 Springs

Twisting a rotational spring will produce an opposing torque. This torque increases as the deformation increases. A simple example of a solid rod torsional spring is shown in Figure 5.9. The angle of rotation is determined by the applied torque, T, the shear modulus, G, the area moment of inertia, J_A, and the length, L, of the rod. The constant parameters can be lumped into a single spring coefficient similar to that used for translational springs.



$$T = \left(\frac{J_A G}{L}\right)\theta \qquad (8)$$

$$T = K_S(\Delta\theta) \qquad (9)$$

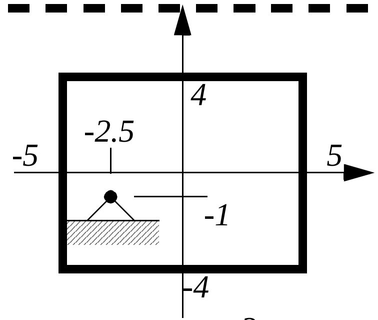
Note: Remember to use radians for these calculations. In fact you are advised to use radians for all calculations. Don't forget to set your calculator to radians also.

Note: This calculation uses the area moment of inertia.

Figure 5.9 A solid torsional spring

The spring constant for a torsional spring will be relatively constant, unless the material is deformed outside the linear elastic range, or the geometry of the spring changes significantly.

When dealing with strength of material properties the area moment of inertia is required. The calculation for the area moment of inertia is similar to that for the mass moment of inertia. An example of calculating the area moment of inertia is shown in Figure 5.10, and based on the previous example in Figure 5.6. The calculations are similar to those for the mass moments of inertia, except for the formulation of the integration elements. Note the difference between the mass moment of inertia and area moment of inertia for the part. The area moment of inertia can be converted to a mass moment of inertia simply by multiplying by the density. Also note the units.



First, the area moment of inertia is calculated about the centroid by integration. All dimensions are in m.

$$I_{xx} = \int_{-4m}^{4m} y^2 dA = \int_{-4m}^{4m} y^2 2(5m)dy = 10m\frac{y^3}{3}\Bigg|_{-4m}^{4m} = 10m\left(\frac{(4m)^3}{3} - \frac{(-4m)^3}{3}\right) = 426.7$$

$$I_{yy} = \int_{-5m}^{5m} x^2 dA = \int_{-5m}^{5m} x^2 2(4m)dx = 8m\frac{x^3}{3}\Bigg|_{-5m}^{5m} = 8m\left(\frac{(5m)^3}{3} - \frac{(-5m)^3}{3}\right) = 666.7m^4$$

$$\tilde{J}_A = I_{xx} + I_{yy} = (426.7 + 666.7)m^4 = 1093.4m^4$$

Next, shift the area moment of inertia from the centroid to the other point of rotation.

$$J_A = \tilde{J}_A + Ar^2$$

$$\therefore \quad = 1093.4m^4 + ((4m-(-4m))(5m-(-5m)))((-1m)^2 + (-2.5m)^2)$$

$$\therefore \quad = 1673m^4$$

Note: The basic definitions for the area moment of inertia are shown to the right.

$$I_{xx} = \int y^2 dA \qquad (8)$$

$$I_{yy} = \int x^2 dA \qquad (9)$$

$$J_A = I_{xx} + I_{yy} \qquad (10)$$

$$J_A = \tilde{J}_A + Ar^2 \qquad (11)$$

Note: You may notice that when the area moment of inertia is multiplied by the density of the material, the mass moment of inertia is the result. Therefore if you have a table of area moments of inertia, multiplying by density will yield the mass moment of inertia. Keep track of units when doing this.

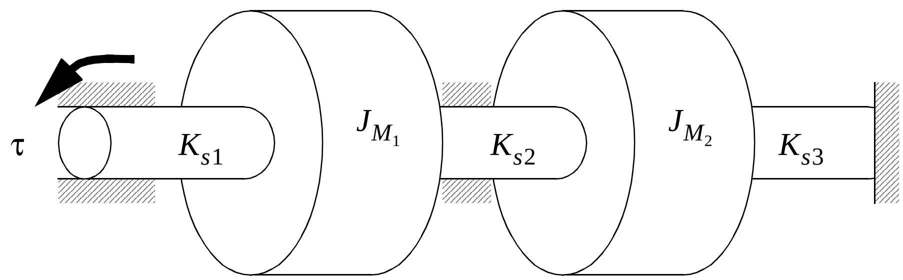
Figure 5.10 Area moment of inertia

For a 1/2" 1020 steel rod that is 1 yard long, find the torsional spring coefficient.

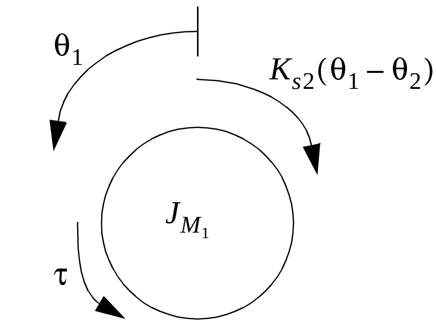
ans. $K_s = 215\frac{Nm}{rad}$

Figure 5.11 Drill problem: Find the torsional spring coefficient

An example problem with torsional springs is shown in Figure 5.12. There are three torsional springs between two rotating masses. The right hand spring is anchored solidly in a wall, and will not move. A torque is applied to the left hand spring. Because the torsional spring is considered massless the torque will be the same at the other end of the spring, at mass J_1. FBDs are drawn for both of the masses, and forces are summed. (Note: the similarity in the methods used for torsional, and for translational springs.) These equations are then rearranged into state variable equations, and finally put in matrix form.



Model the system above assuming that the center shaft is a torsional spring, and that a torque is applied to the leftmost disk. Leave the results in state variable form.

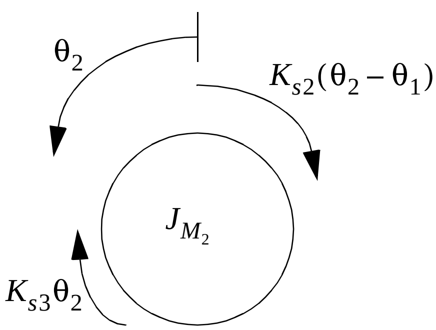


$$+\quad \sum M = \tau - K_{s2}(\theta_1 - \theta_2) = J_{M_1}\ddot{\theta}_1$$

$$J_{M_1}\ddot{\theta}_1 = -K_{s2}\theta_1 + K_{s2}\theta_2 + \tau$$

$$\dot{\theta}_1 = \omega_1 \qquad (1)$$

$$\dot{\omega}_1 = \left(\frac{-K_{s2}}{J_{M_1}}\right)\theta_1 + \left(\frac{K_{s2}}{J_{M_1}}\right)\theta_2 + \tau \qquad (2)$$



$$+\quad \sum M = -K_{s2}(\theta_2 - \theta_1) - K_{s3}\theta_2 = J_{M_2}\ddot{\theta}_2$$

$$\ddot{\theta}_2 = \left(\frac{-K_{s3} - K_{s2}}{J_{M_2}}\right)\theta_2 + \left(\frac{K_{s2}}{J_{M_2}}\right)\theta_1$$

$$\dot{\theta}_2 = \omega_2 \qquad (3)$$

$$\dot{\omega}_2 = \left(\frac{-K_{s3} - K_{s2}}{J_{M_2}}\right)\theta_2 + \left(\frac{K_{s2}}{J_{M_2}}\right)\theta_1 \qquad (4)$$

$$\frac{d}{dt}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K_{s2}}{J_{M_1}} & 0 & \frac{K_{s2}}{J_{M_1}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K_{s2}}{J_{M_2}} & 0 & \frac{-K_{s3} - K_{s2}}{J_{M_2}} & 0 \end{bmatrix}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} + \begin{bmatrix}0\\ \tau\\ 0\\ 0\end{bmatrix}$$

Figure 5.12 A rotational spring example

5.2.3 Damping

Rotational damping is normally caused by viscous fluids, such as oils, used for lubrication. It opposes angular velocity with the relationships shown in Figure 5.13. The first equation is used for a system with one rotating and one stationary part. The second equation is used for damping between two rotating parts.

$$T = K_d \omega$$

$$T = K_d(\omega_1 - \omega_2)$$

Figure 5.13 The rotational damping equation

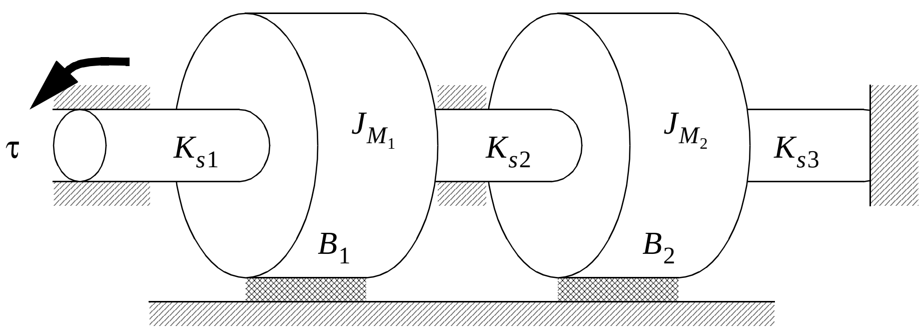
If a wheel (J_M=5kg m^2) is turning at 150 rpm and the damping coefficient is 1Nms/rad, what is the deceleration?

ans.

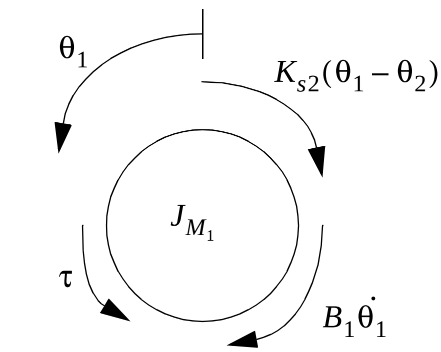
$$\ddot{\theta} = -3.141 \frac{rad}{s^2}$$

Figure 5.14 Drill problem: Find the deceleration

The example in Figure 5.12 is extended to include damping in Figure 5.15. The primary addition from the previous example is the addition of the damping forces to the FBDs. In this case the damping coefficients are indicated with 'B', but 'Kd' could have also been used. The state equations were developed in matrix form. Visual comparison of the final matrices in this and the previous example reveal that the damping terms are the only addition.



Model the system above assuming that the center shaft is a torsional spring, and that a torque is applied to the leftmost disk. Leave the results in state variable form.

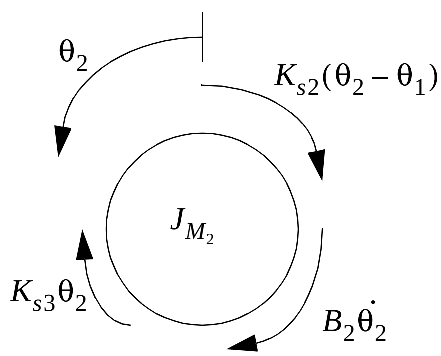


$$+\sum M = \tau - K_{s2}(\theta_1 - \theta_2) - B_1\dot{\theta}_1 = J_{M_1}\ddot{\theta}_1$$

$$J_{M_1}\ddot{\theta}_1 = -B_1\dot{\theta}_1 - K_{s2}\theta_1 + K_{s2}\theta_2 + \tau$$

$$\dot{\theta}_1 = \omega_1 \qquad (1)$$

$$\dot{\omega}_1 = \left(\frac{-B_1}{J_{M_1}}\right)\omega_1 + \left(\frac{-K_{s2}}{J_{M_1}}\right)\theta_1 + \left(\frac{K_{s2}}{J_{M_1}}\right)\theta_2 + \frac{\tau}{J_{M_1}} \qquad (2)$$



$$+\sum M = -K_{s2}(\theta_2 - \theta_1) - B_2\dot{\theta}_2 - K_{s3}\theta_2 = J_{M_2}\ddot{\theta}_2$$

$$\ddot{\theta}_2 = \left(\frac{-B_2}{J_{M_2}}\right)\dot{\theta}_2 + \left(\frac{-K_{s3} - K_{s2}}{J_{M_2}}\right)\theta_2 + \left(\frac{K_{s2}}{J_{M_2}}\right)\theta_1$$

$$\dot{\theta}_2 = \omega_2 \qquad (3)$$

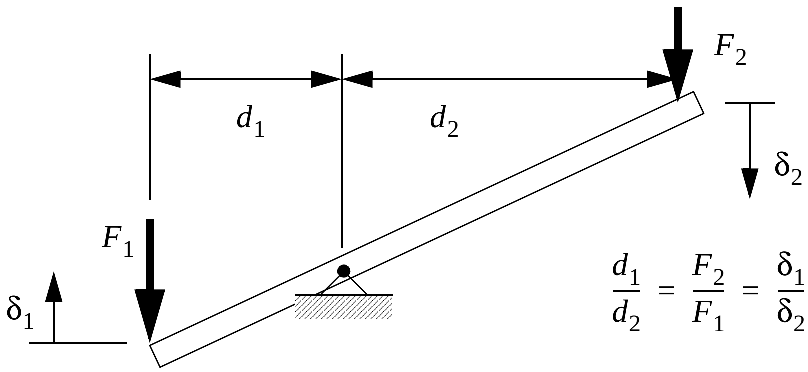
$$\dot{\omega}_2 = \left(\frac{-B_2}{J_{M_2}}\right)\omega_2 + \left(\frac{-K_{s3} - K_{s2}}{J_{M_2}}\right)\theta_2 + \left(\frac{K_{s2}}{J_{M_2}}\right)\theta_1 \qquad (4)$$

$$\frac{d}{dt}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K_{s2}}{J_{M_1}} & \frac{-B_1}{J_{M_1}} & \frac{K_{s2}}{J_{M_1}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K_{s2}}{J_{M_2}} & 0 & \frac{-K_{s3} - K_{s2}}{J_{M_2}} & \frac{-B_2}{J_{M_2}} \end{bmatrix}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\tau}{J_{M_1}} \\ 0 \\ 0 \end{bmatrix}$$

Figure 5.15 A System Example

5.2.4 Levers

The lever shown in Figure 5.16 can be used to amplify forces or motion. Although theoretically a lever arm could rotate fully, it typically has a limited range of motion. The amplification is determined by the ratio of arm lengths to the left and right of the center.



$$\frac{d_1}{d_2} = \frac{F_2}{F_1} = \frac{\delta_1}{\delta_2}$$

Note: As the lever rotates the length ratio will be maintained because of similar triangles. This allows the lever to work over a range of angles. The lever above would become ineffective if it was vertical.

Note: The tip deflection can be related to the angle of rotation of the lever if the angle of rotation is small. $\theta = \frac{\delta_1}{d_1} = \frac{\delta_2}{d_2}$

Figure 5.16 Force transmission with a level

Given a lever set to lift a 1000 kg rock - if the lever is 2m long and the distance from the fulcrum to the rock is 10cm, how much force is required to lift it?

ans.
$F = 516.3N$

Figure 5.17 Drill problem: Find the required force

5.2.5 Gears and Belts

While levers amplify forces and motions over limited ranges of motion, gears can rotate indefinitely. Some of the basic gear forms are listed below.

Spur - Round gears with teeth parallel to the rotational axis.
Rack - A straight gear (used with a small round gear called a pinion).
Helical - The teeth follow a helix around the rotational axis.
Bevel - The gear has a conical shape, allowing forces to be transmitted at angles.

Gear teeth are carefully designed so that they will *mesh* smoothly as the gears rotate. The forces on gears acts at a tangential distance from the center of rotation called the *pitch diameter*. The ratio of motions and forces through a pair of gears is proportional to their radii, as shown in Figure 5.18. The number of teeth on a gear is proportional to the diameter. The gear ratio is used to measure the relative rotations of the shafts. For example a gear ratio of 20:1 would mean the input shaft of the gear box would have to rotate 20 times for the output shaft to rotate once.

$$T_1 = F_c r_1 \qquad T_2 = -F_c r_2 \qquad \frac{n_1}{r_1} = \frac{n_2}{r_2} \qquad \frac{-T_1}{T_2} = \frac{r_1}{r_2} = \frac{n_1}{n_2}$$

$$V_c = \omega_1 r_1 = -\omega_2 r_2 \qquad \frac{r_2}{r_1} = \frac{-\omega_1}{\omega_2} = \frac{-\alpha_1}{\alpha_2} = \frac{-\Delta\theta_1}{\Delta\theta_2} = \frac{n_2}{n_1}$$

where,
n = number of teeth on respective gears
r = radii of respective gears
Fc = force of contact between gear teeth
Vc = tangential velocity of gear teeth
T = torques on gears

Figure 5.18 Basic Gear Relationships

For lower gear ratios a simple gear box with two gears can be constructed. For higher gear ratios more gears can be added. To do this, compound gear sets are required. In a compound gear set two or more gears are connected on a single shaft, as shown in Figure 5.19. In this example the gear ratio on the left is 4:1, and the ratio for the set on the right is 4:1. Together they give a gear ratio of 16:1.

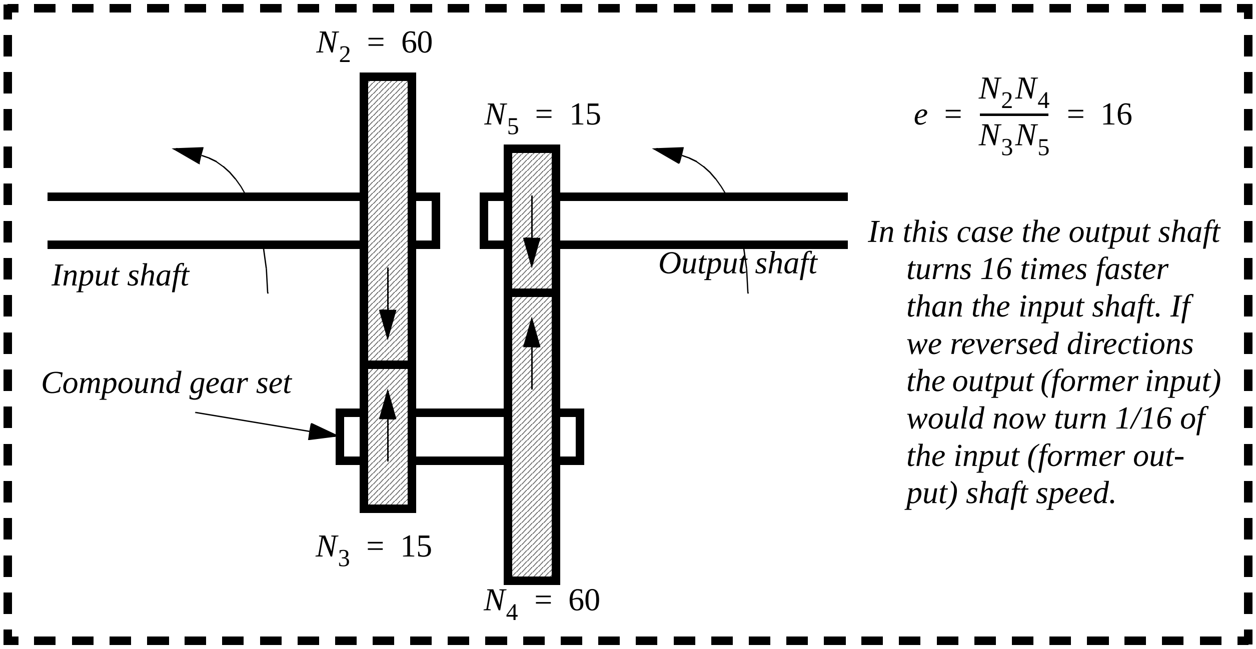
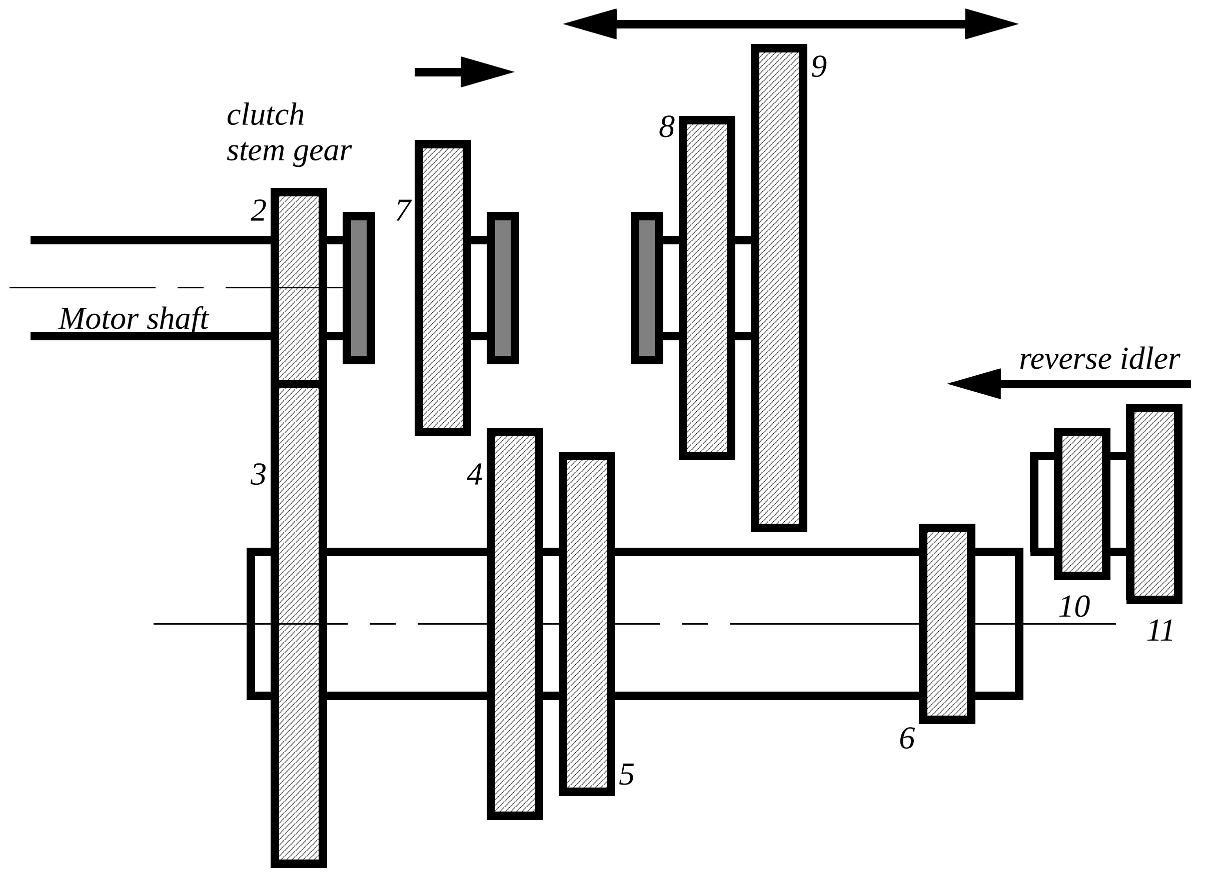


Figure 5.19 A compound gear set

A manual transmission is shown in Figure 5.20. In the transmission the gear ratio is changed by sliding (left-right) some of the gears to change the sequence of gears transmitting the force. Notice that when in reverse an additional compound gear set is added to reverse the direction of rotation.



Speed (gear)	*Gear Train*
1	*2-3-6-9*
2	*2-3-5-8*
3	*2-3-4-7*
4	*bypass gear train*
reverse	*2-3-6-10-11-9*

In this manual transmission the gear shifter will move the gears in and out of contact. At this point all of the needed gears will be meshed and turning. The final step is to engage the last gear in the gear train with the clutch (plate) and this couples the gears to the wheels.

Figure 5.20 A manual transmission

Rack and pinion gear sets are used for converting rotation to translation. A rack is a long straight gear that is driven by a small mating gear called a pinion. The basic relationships are shown in Figure 5.21.

$$T = Fr \qquad V_c = \omega r \qquad \Delta l = r\Delta\theta$$

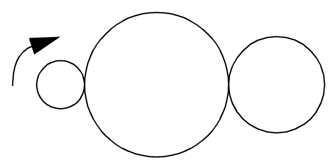
where,

r = radius of pinion
F = force of contact between gear teeth
Vc = tangential velocity of gear teeth and velocity of rack
T = torque on pinion

Figure 5.21 Relationships for a rack and pinion gear set

Belt based systems can be analyzed with methods similar to gears (with the exception of teeth). A belt wound around a drum will act like a rack and pinion gear pair. A belt around two or more pulleys will act like gears.

A gear train has an input gear with 20 teeth, a center gear that has 100 teeth, and an output gear that has 40 teeth. If the input shaft is rotating at 5 rad/sec what is the rotation speed of the output shaft?



What if the center gear is removed?

ans.

case 1: $\omega_3 = 2.5\frac{rad}{s}$

case 2: $\omega_3 = -2.5\frac{rad}{s}$

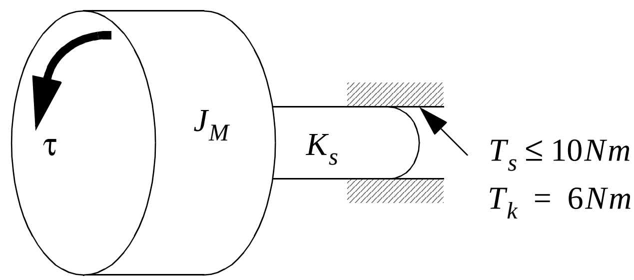
Figure 5.22 Drill problem: Find the gear speed

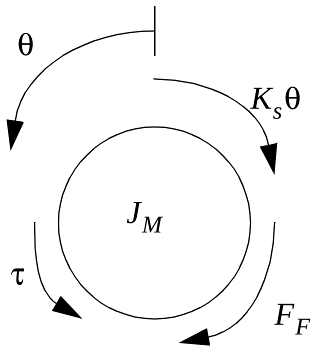
5.2.6 Friction

Friction between rotating components is a major source of inefficiency in machines. It is the result of contact surface materials and geometries. Calculating friction values in rotating systems is more difficult than translating systems. Normally rotational friction will be given as static and kinetic friction torques.

An example problem with rotational friction is shown in Figure 5.23. Basically these problems require that the model be analyzed as if the friction surface is fixed. If the friction force exceeds the maximum static friction the mechanism is then analyzed using the dynamic friction torque. There is friction between the shaft and the hole in the wall. The friction force is left as a variable for the derivation of the state equations. The friction value must be calculated using the appropriate state equation. The result of this calculation and the previous static or dynamic condition is then used to determine the new friction value.

Model the system and consider the static and kinetic friction forces on the shaft on the right hand side.





$$+ \quad \sum M = \tau - K_s\theta - T_F = J_M\ddot{\theta}$$

$$J_M\ddot{\theta} = \tau - K_s\theta - T_F \tag{1}$$

$$\dot{\theta} = \omega \tag{2}$$

$$\dot{\omega} = \left(\frac{\tau - T_F}{J_M}\right) + \left(\frac{K_s}{J_M}\right)\theta \tag{3}$$

Next, the torque force must be calculated, and then used to determine the new torque force.

$$J_M\dot{\omega} = \tau - K_s\theta - T_{test}$$
$$T_{test} = \tau - K_s\theta - J_M\dot{\omega} \tag{4}$$

cases:

Not slipping previously

$T_{test} \leq 10Nm \qquad T_F = T_{test}$

$T_{test} > 10Nm \qquad T_F = 6Nm$

Slipping previously

$T_{test} < 6Nm \qquad T_F = T_{test}$

$T_{test} \geq 6Nm \qquad T_F = 6Nm$

Figure 5.23 A friction system example

The friction example in Figure 5.23 can be analyzed using the C program in Figure 5.24. For the purposes of the example some component values are selected and the system is assumed to be at rest initially. The program loops to integrate the state equations. Each loop the friction conditions are checked and then used for a first-order solution to the state equations.

```
int main(){
        double          h = 0.1, /* time step */
                        theta, w, /* the state variables */
                        acceleration, /* the acceleration */
                        TF, /* friction force */
                        Ttest, /* the friction test force */
                        J = 10, /* the moment of inertia (I picked the value) */
                        tau = 5, /* the applied torque (I picked the value) */
                        Ks = 10; /* the spring constant (I picked the value) */
        int             slip = 0; /* the system starts with no slip */
        FILE            *fp;

        theta = 0; w = 0;/* the initial conditions - starting at rest here */
        TF = 0.0; /* set the initial friction to 0.0; */
        acceleration = 0.0;/* set the initial acceleration to zero also */
        if( ( fp = fopen("out.txt", "w")) != NULL){/* open a file to write the results */
                for( t = 0.0; t < 10.0; t += h ){/* loop */
                        Ttest = tau - Ks*theta - J*acceleration;
                        if(slip == 0){ /* not slipping */
                                if(Ttest >= 10){
                                        TF = 6; slip = 1;
                                } else {
                                        TF = Ttest;
                                }
                        } else { /* slipping */
                                if(Ttest < 6){ TF = Ttest; slip = 0;
                                } else {TF = 6;}
                        }
                        acceleration = (tau - TF + Ks*theta) / J;
                        w = w + h * acceleration;
                        theta = theta + h * w;
                        fprintf(fp, "%f, %f, %f\n", t, theta, w);
                }
        }
        fclose(fp);
}
```

Figure 5.24 A C program for the friction example in Figure 5.23

5.2.7 Permanent Magnet Electric Motors

DC motors create a torque between the rotor (inside) and stator (outside) that is related to the applied voltage or current. In a permanent magnet motor there are magnets mounted on the stator, while the rotor consists of wound coils. When a voltage is applied to the coils the motor will accelerate. The differential equation for a motor is shown in Figure 5.25, and will be derived in a later chapter. The value of the constant 'K' is a function of the motor design and will remain fixed. The value of the coil resistance 'R' can be directly measured from the motor. The moment of inertia 'J' should include the motor shaft, but when a load is added this should be added to the value of 'J'.

$$\therefore \left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{JR}\right) = V_s\left(\frac{K}{JR}\right) - \frac{T_{load}}{J_M}$$

where,

ω = *the angular velocity of the motor*

K = *the motor speed constant*

J_M = *the moment of inertia of the motor and attached loads*

R = *the resistance of the motor coils*

T_{load} = *a torque applied to a motor shaft*

Figure 5.25 Model of a permanent magnet DC motor

The speed response of a permanent magnet DC motor is first-order. The steady-state velocity will be a straight line function of the torque applied to the motor, as shown in Figure 5.26. In addition the line shifts outwards as the voltage applied to the motor increases.

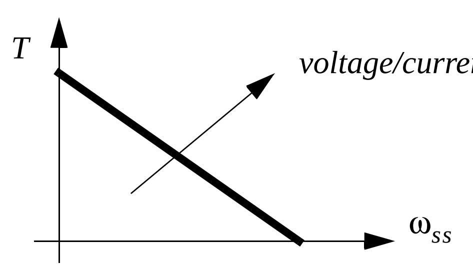


Figure 5.26 Torque speed curve for a permanent magnet DC motor

5.3 OTHER TOPICS

The energy and power relationships for rotational components are given in Figure 5.27. These can be useful when designing a system that will store and release energy.

$$E = E_K + E_P \qquad (5)$$

$$E_K = J_M \omega^2 \qquad (6)$$

$$E_P = T\theta \qquad (7)$$

$$P = T\omega \qquad (8)$$

Figure 5.27 Energy and power relations for rotation

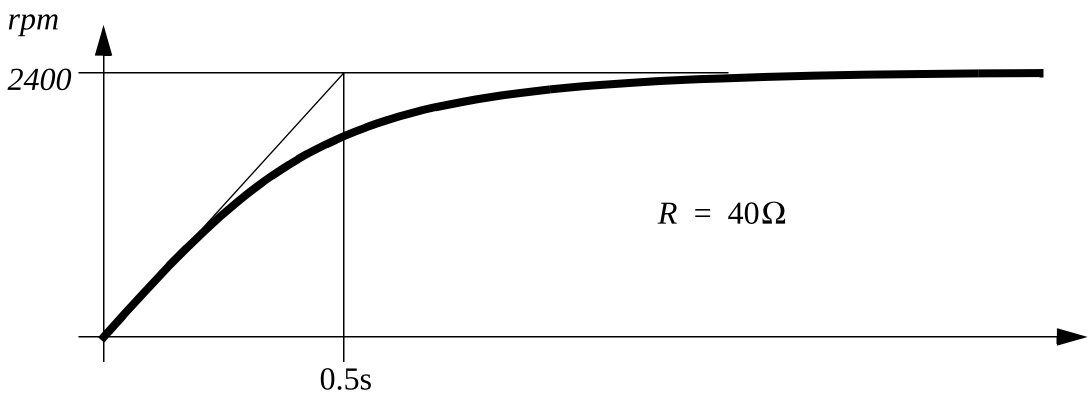
Note: The units for various rotational quantities are listed to the right. They may be used to check equations by doing a unit balance. The unit 'rad' should be ignored as it appears/disappears sporadically.

coefficient	units
K_s	$\frac{Nm}{rad}$
K_d, B	$\frac{Nms}{rad}$
J_M	Kgm^2

5.4 DESIGN CASE

A large machine is to be driven by a permanent magnet electric motor. A 20:1 gear box is used to reduce the speed and increase the torque of the motor. The motor drives a 10000kg mass in translation through a rack and pinion gear set. The pinion has a pitch diameter of 6 inches. A 10 foot long shaft is required between the gear box and the rack and pinion set. The mass moves on rails with static and dynamic coefficients of friction of 0.2 and 0.1 respectively. We want to select a shaft diameter that will keep the system critically damped when a voltage step input of 20V is applied to the motor.

To begin the analysis the velocity curve in Figure 5.28 was obtained experimentally by applying a voltage of 15V to the motor with no load attached. In addition the resistance of the motor coils was measured and found to be 40 ohms. The steady-state speed and time constant were used to determine the constants for the motor.



$$\left(\frac{d}{dt}\right)\omega_m + \omega_m\left(\frac{K^2}{J_M R}\right) = V_s\left(\frac{K}{J_M R}\right) - \frac{T_{load}}{J_M}$$

The steady-state velocity can be used to find the value of K.

$$(0) + \left(2400\frac{rot}{min}\right)\left(\frac{K^2}{J_M R}\right) = 15V\left(\frac{K}{J_M R}\right) - (0)$$

$$\left(2400\frac{rot}{min}\frac{1min}{60s}\frac{2\pi rad}{1rot}\right)(K) = 15V$$

$$K = \frac{15V}{120\pi rads^{-1}} = 39.8\times10^{-3}\frac{Vs}{rad}$$

The time constant can be used to find the remaining parameters.

$$\frac{K^2}{J_M R} = \frac{1}{0.5s} = 2s^{-1}$$

$$J = \frac{\left(39.8\times10^{-3}\frac{Vs}{rad}\right)^2}{(40\Omega)(2s^{-1})} = 0.198005\times10^{-4} = 19.8\times10^{-6}Kgm^2$$

$$\left(\frac{d}{dt}\right)\omega_m + \omega_m 2s^{-1} = V_s(50.3V^{-1}s^{-2}rad) - \frac{T_{load}}{19.8\times10^{-6}Kgm^2}$$

$$\theta_m' = \omega_m \qquad (1)$$

$$\omega_m' = V_s 50.3V^{-1}s^{-2}rad - \omega_m 2s^{-1} - 50505Kg^{-1}m^{-2}T_{load} \qquad (2)$$

Figure 5.28 Motor speed curve and the derived differential equation

The remaining equations describing the system are developed in Figure 5.29. These calculations are done with the assumption that the inertial effects of the gears and other components are insignificant.

The long shaft must now be analyzed. This will require that angles at both ends be defined, and the shaft be considered as a spring.

$\theta_{gear}, \omega_{gear}$ = *angular position and velocity of the shaft at the gear box*

$\theta_{pinion}, \omega_{pinion}$ = *angular position and velocity of the shaft at the pinion*

$$\theta_{gear} = \frac{1}{20}\theta_m \qquad \omega_{gear} = \frac{1}{20}\omega_m$$

$$T_{shaft} = K_s(\theta_{gear} - \theta_{pinion})$$

The rotation of the pinion is related to the displacement of the rack through the circumferential travel. This ratio can also be used to find the force applied to the mass.

$$x_{mass} = \theta_{pinion}\pi 6in$$

$$T_{shaft} = F_{mass}\left(\frac{6in}{2}\right)$$

$$K_s(\theta_{gear} - \theta_{pinion}) = F_{mass}\left(\frac{6in}{2}\right)$$

$$\sum F_{mass} = F_{mass} = M_{mass}\ddot{x}_{mass}$$

$$\frac{K_s(\theta_{gear} - \theta_{pinion})}{\left(\frac{6in}{2}\right)} = M_{mass}\ddot{\theta}_{pinion}\pi 6in$$

$$\ddot{\theta}_{pinion} = (\theta_{gear} - \theta_{pinion})1.768 \times 10^{-6} in^{-2} Kg^{-1} K_s$$

$$\ddot{\theta}_{pinion} = \left(\frac{1}{20}\theta_m - \theta_{pinion}\right)1.768 \times 10^{-6} in^{-2} Kg^{-1} K_s\left(\frac{0.0254in}{1.0m}\right)^2$$

$$\ddot{\theta}_{pinion} = \left(\frac{1}{20}\theta_m - \theta_{pinion}\right)(1.141 \times 10^{-9}) m^{-2} Kg^{-1} K_s$$

$$\dot{\theta}_{pinion} = \omega_{pinion} \qquad (3)$$

$$\dot{\omega}_{pinion} = 57.1 \times 10^{-12} m^{-2} Kg^{-1} K_s \theta_m - 1.141 \times 10^{-9} m^{-2} Kg^{-1} K_s \theta_{pinion} \qquad (4)$$

Figure 5.29 Additional equations to model the machine

If the gear box is assumed to have relatively small moment of inertia, then we can say that the torque load on the motor is equal to the torque in the shaft. This then allows the equation for the motor shaft to be put into a useful form, as shown in Figure 5.30. Having this differential equation now allows the numerical analysis to proceed. The analysis involves iteratively solving the equations and determining the point at which the system begins to overshoot, indicating critical damping.

The Tload term is eliminated from equation (2)

$$\dot{\omega}_m = V_s 50.3V^{-1}s^{-2}rad - \omega_m 2s^{-1} - 50505Kg^{-1}m^{-2}K_s(\theta_{gear} - \theta_{pinion})$$

$$\dot{\omega}_m = V_s 50.3V^{-1}s^{-2}rad - \omega_m 2s^{-1} - 50505Kg^{-1}m^{-2}K_s\left(\frac{1}{20}\theta_m - \theta_{pinion}\right)$$

$$\dot{\omega}_m = (V_s 50.3V^{-1}s^{-2}rad) + \theta_{pinion}(50505Kg^{-1}m^{-2}K_s) + \omega_m(-2s^{-1}) + \theta_m(-2525Kg^{-1}m^{-2}K_s)$$

The state equations can then be put in matrix form for clarity. The units will be eliminated for brevity, but acknowledging that they are consistent.

$$\frac{d}{dt}\begin{bmatrix} \theta_m \\ \omega_m \\ \theta_{pinion} \\ \omega_{pinion} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2525K_s & -2 & 50505K_s & 0 \\ 0 & 0 & 0 & 1 \\ 57.1 \times 10^{-12}K_s & 0 & -1.141 \times 10^{-9}K_s & 0 \end{bmatrix}\begin{bmatrix} \theta_m \\ \omega_m \\ \theta_{pinion} \\ \omega_{pinion} \end{bmatrix} + \begin{bmatrix} 0 \\ V_s 50.3 \\ 0 \\ 0 \end{bmatrix}$$

The state equations for the system are then analyzed using a computer for the parameters below to find the Ks value that gives a response that approximates critical damping for a step input from 0 to 10V.

Ks (rad/Nm)	*Overshoot (rad)*
100	

Figure 5.30 Numerical analysis of system response

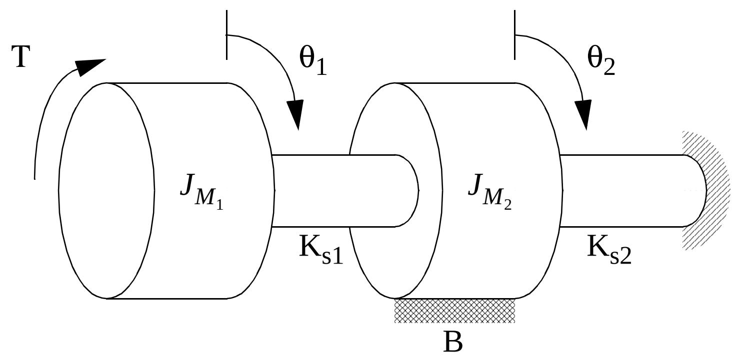
These results indicate that a spring value of XXX is required to have the system behave as if it is critically damped. (Note: Clearly this system is not second order, but in the absence of another characteristics we approximate it as second order.)

5.5 SUMMARY

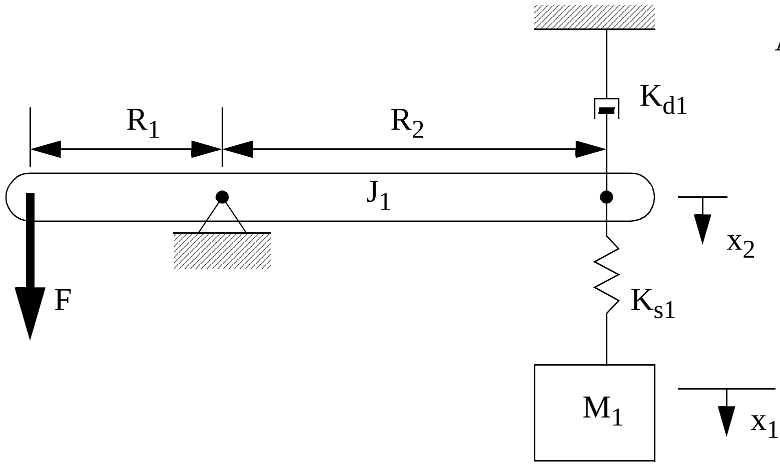
- The basic equations of motion were discussed.
- Mass and area moment of inertia are used for inertia and springs.
- Rotational dampers and springs.
- A design case was presented.

5.6 PRACTICE PROBLEMS

1. Draw the FBDs and write the differential equations for the mechanism below. The right most shaft is fixed in a wall.

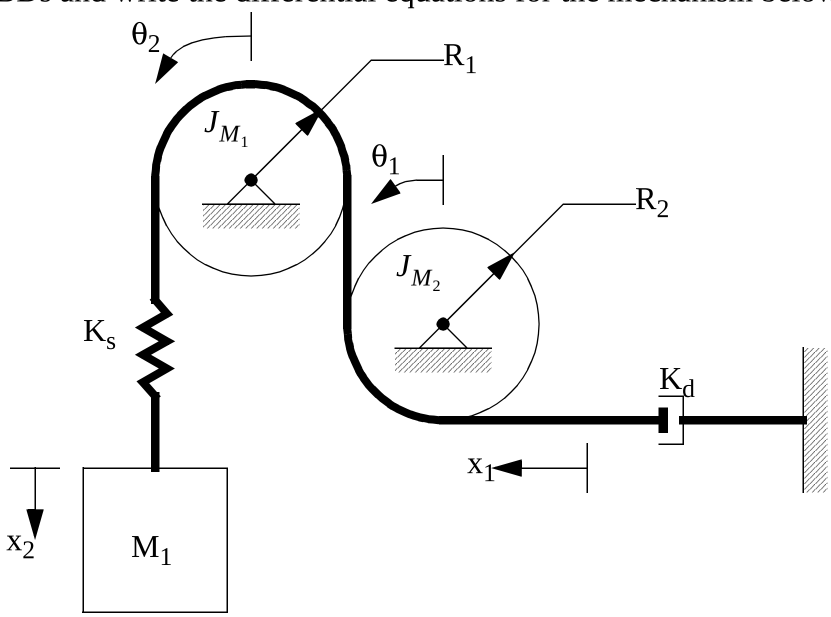


2. For the system pictured below a) write the differential equations (assume small angular deflections) and b) put the equations in state variable form.



A lever arm has a force on one side, and a spring damper combination on the other side with a suspended mass.

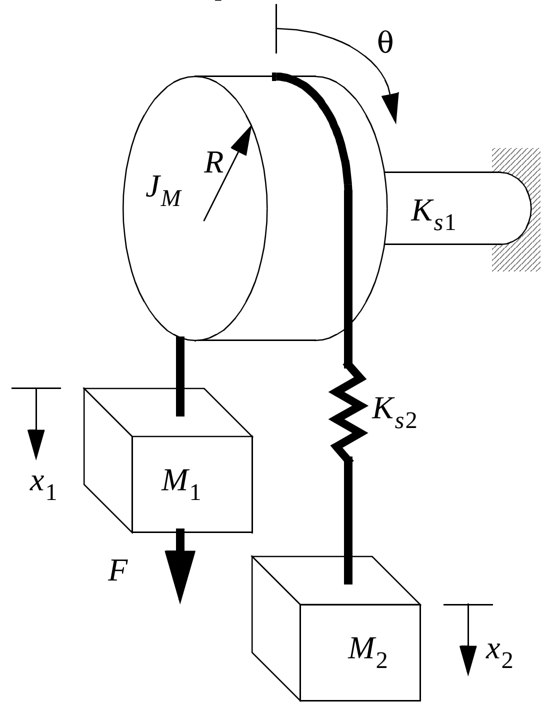
3. Draw the FBDs and write the differential equations for the mechanism below.



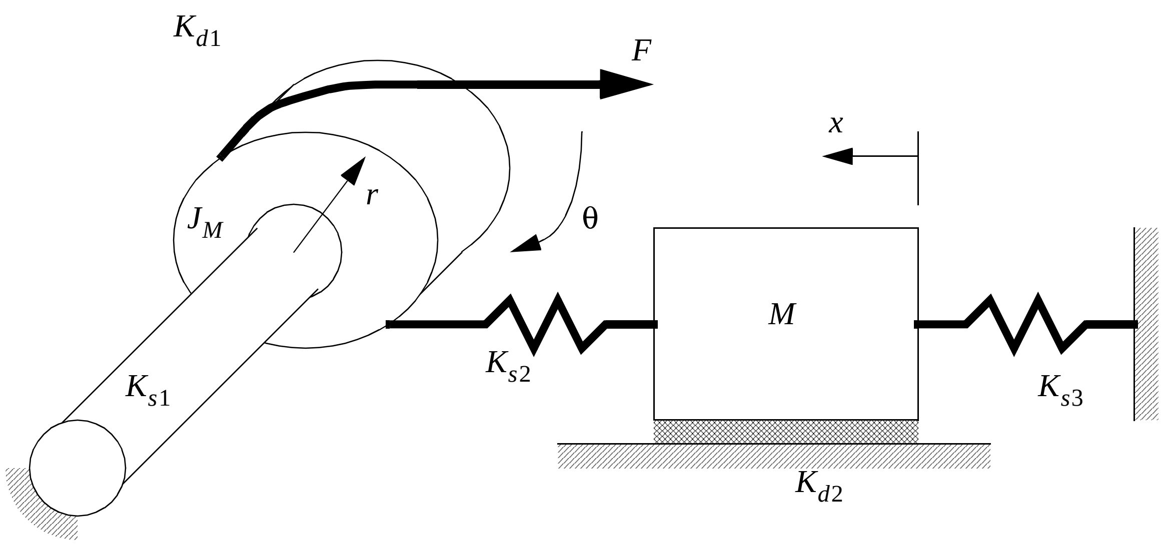
4. The system below consists of two masses hanging by a cable over mass 'J'. There is a spring in the cable near M2. The cable doesn't slip on 'J'.

a) Derive the differential equations for the following system.

b) Convert the differential equations to state variable equations

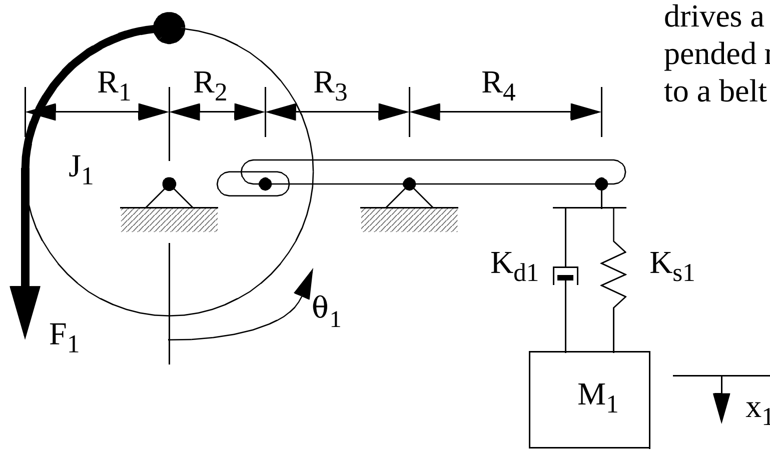


5. Write the state equations for the system to relate the applied force 'F' to the displacement 'x'. Note that the rotating mass also experiences a rotational damping force indicated with Kd1



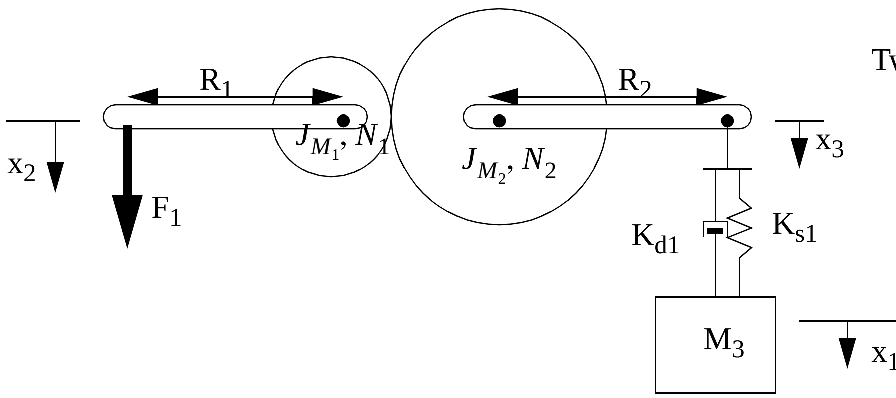
6. For the system pictured below a) write the differential equations (assume small angular deflec-

tions) and b) put the equations in state variable form.



A round drum with a slot. The slot drives a lever arm with a suspended mass. A force is applied to a belt over the drum.

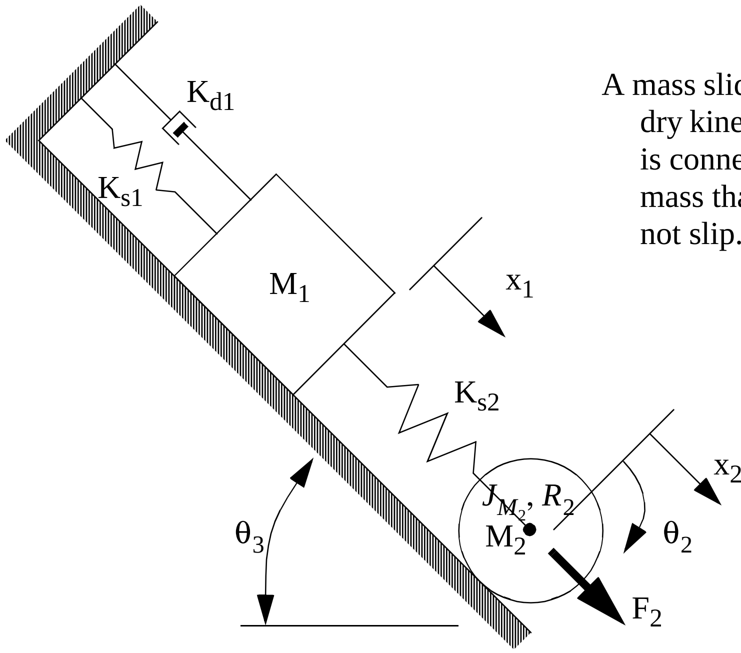
7. For the system pictured below a) write the differential equations (assume small angular deflections) and b) put the equations in state variable form.



Two gears with fixed centers of rotation and lever arms.

8. For the system pictured below a) write the differential equations (assume small angular deflec-

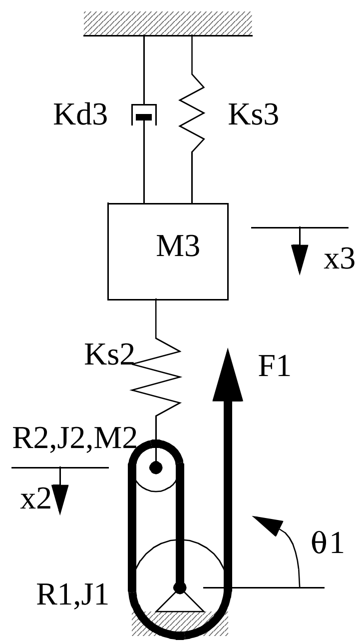
tions) and b) put the equations in state variable form.



A mass slides on a plane with dry kinetic friction (0.3). It is connected to a round mass that rolls and does not slip.

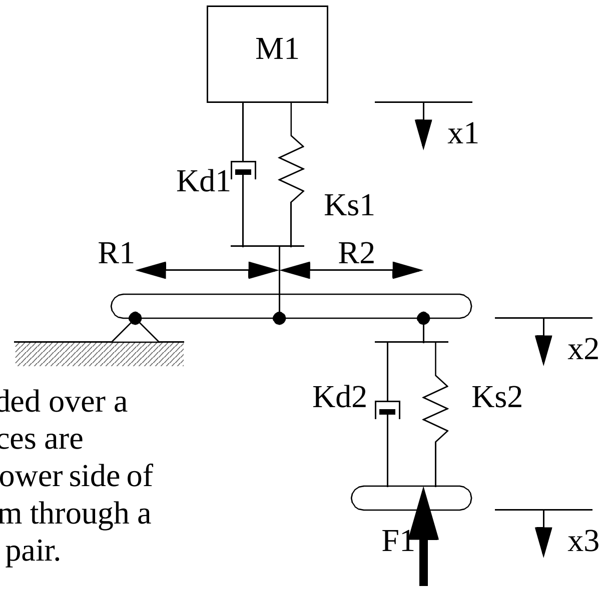
9. For the system pictured below a) write the differential equations (assume small angular deflections) and b) put the equations in state variable form.

A pulley system has the bottom pulley anchored. A mass is hung in the middle of the arrangement with springs and dampers on either side. Assume that the cable is always tight.



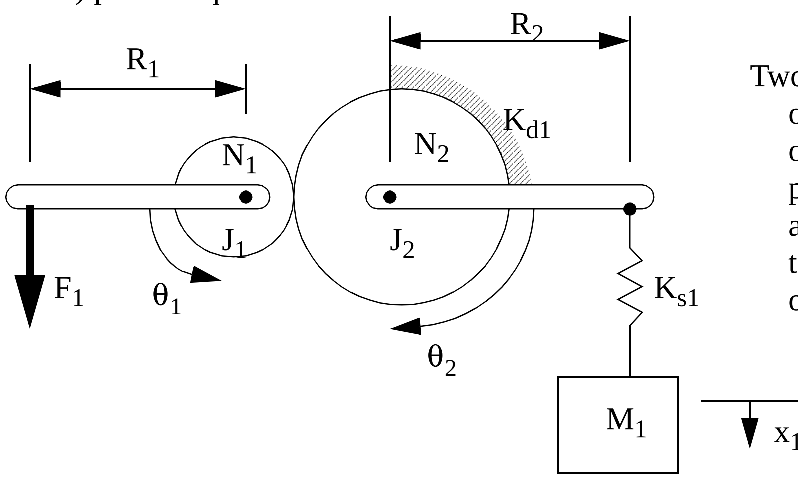
10. For the system pictured below a) write the differential equations (assume small angular

deflections) and b) put the equations in state variable form.



A mass is suspended over a lever arm. Forces are applied to the lower side of the moment arm through a spring damper pair.

11. For the system pictured below a) write the differential equations (assume small angular deflections) and b) put the equations in state variable form.



Two gears have a force on one side, and a mass on the other, both suspended from moment arms. There is a rotational damping on one of the gears.

12. Find the polar moments of inertia of area and mass for a round cross section with known radius and mass per unit area. How are they related?

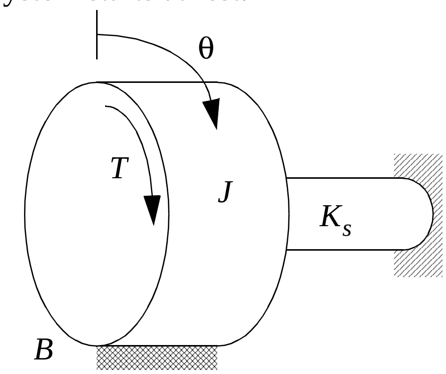
13. The rotational spring is connected between a mass ‘J’, and the wall where it is rigidly held. The mass has an applied torque ‘T’, and also experiences damping ‘B’.

a) Derive the differential equation for the rotational system shown.

b) Put the equation in state variable form (using variables) and then plot the position (not velocity) as a function of time for the first 5 seconds with your calculator using the parameters below. Assume the system starts at rest.

$$K_s = 10\frac{Nm}{rad} \qquad B = 1\frac{Nms}{rad}$$

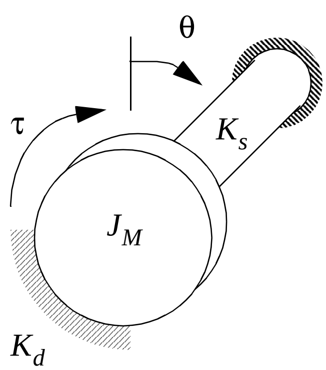
$$J_M = 1Kgm^2 \qquad T = 10Nm$$



c) A differential equation for the rotating mass with a spring and damper is given below. Solve the differential equation to get a function of time. Assume the system starts at rest.

$$\theta'' + (1s^{-1})\theta' + (10s^{-2})\theta = 10s^{-2}$$

14. Find the response as a function of time (i.e. solve the differential equation to get a function of time.). Assume the system starts undeflected and at rest.



$$\tau = 10Nm$$

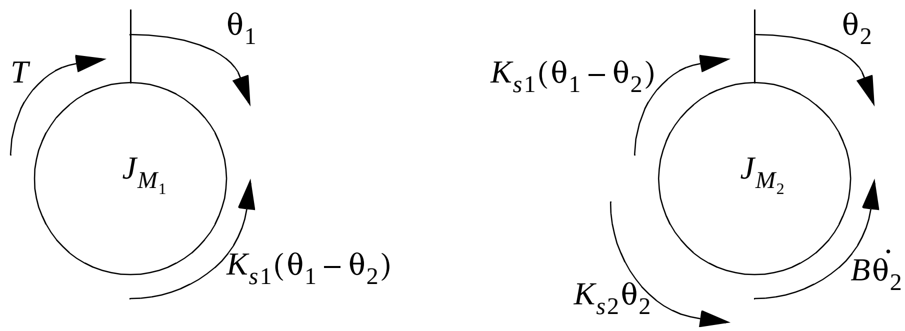
$$J_M = 1Kgm^2$$

$$K_d = 3\frac{Ns}{m}$$

$$K_s = 9\frac{N}{m}$$

5.7 PRACTICE PROBLEM SOLUTIONS

1.



$$+\curvearrowright \quad \sum M_1 = T - K_{s1}(\theta_1 - \theta_2) = J_{M_1}\ddot{\theta}_1$$

$$\ddot{\theta}_1 + \theta_1\left(\frac{K_{s1}}{J_{M_1}}\right) + \theta_2\left(\frac{-K_{s1}}{J_{M_1}}\right) = \frac{T}{J_{M_1}}$$

$$+\curvearrowright \quad \sum M_2 = K_{s1}(\theta_1 - \theta_2) - B\dot{\theta}_2 - K_{s2}\theta_2 = J_{M_2}\ddot{\theta}_2$$

$$\ddot{\theta}_2 + \dot{\theta}_2\left(\frac{B}{J_{M_2}}\right) + \theta_2\left(\frac{K_{s1} + K_{s2}}{J_{M_2}}\right) + \theta_1\left(\frac{-K_{s1}}{J_{M_2}}\right) = 0$$

2.

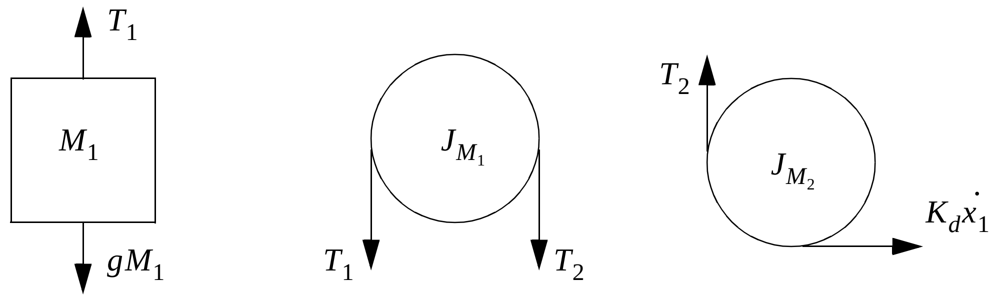
$$\dot{x}_1 = v_1$$

$$\dot{x}_1 = x_1\left(\frac{-K_{s1}}{M_1}\right) + x_2\left(\frac{K_{s1}}{M_1}\right) + g$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_2 = v_1\left(\frac{-R_2^2 K_{d1}}{J_1}\right) + x_1\left(\frac{-R_2^2 K_{s1}}{J_1}\right) + x_2\left(\frac{R_2^2 K_{s1}}{J_1}\right) + \frac{-FR_1}{J_1}$$

3.



if T1,T2,Kdx1 > 0 $\quad \theta_1 = \frac{-x_1}{R_2} \qquad \theta_2 = \frac{x_1}{R_1} \qquad T_1 = K_s(x_2 - x_1)$

$$+ \quad \sum F_y = T_1 - gM_1 = -M_1\ddot{x}_2$$

$$\ddot{x}_2 = g - \frac{T_1}{M_1}$$

$$+ \quad \sum M_1 = -T_1R_1 + T_2R_1 = -J_{M_1}\ddot{\theta}_2$$

$$\ddot{\theta}_2 = \frac{T_1R_1 - T_2R_1}{J_{M_1}}$$

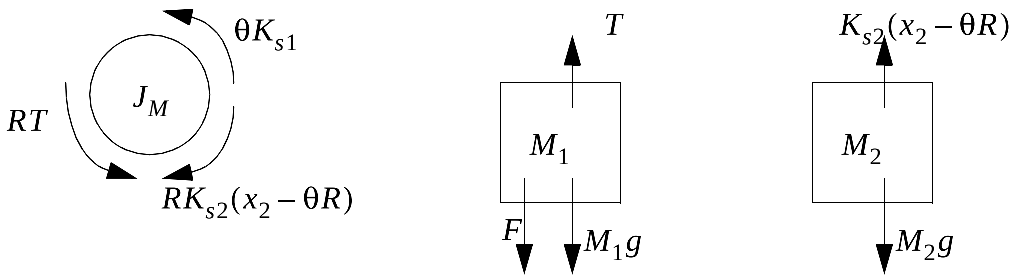
$$+ \quad \sum M_2 = T_2R_2 - R_2K_d\dot{x}_1 = -J_{M_2}\ddot{\theta}_1$$

$$\ddot{\theta}_1 + \dot{x}_1\left(\frac{-R_2K_d}{J_{M_2}}\right) = \frac{T_2R_2}{J_{M_2}}$$

6 equations, 6 unknowns

4.

a)



if(T < 0) T=0

if(T >= 0) $R\theta = -x_1$

$$\sum F_{M1} = T - M_1 g - F = -M_1 \ddot{x}_1$$

$$T = -M_1 \ddot{x}_1 + M_1 g + F = M_1 R\ddot{\theta} + M_1 g + F$$

$$\sum M_J = -RT - \theta K_{s1} + RK_{s2}(x_2 - \theta R) = J_M \ddot{\theta}$$

$$-R(M_1 g + F) - \theta(K_{s1} + R^2 K_{s2}) + (RK_{s2})x_2 = (J_M + R^2 M_1)\ddot{\theta}$$

$$\ddot{\theta} + \theta\left(\frac{K_{s1} + R^2 K_{s2}}{J_M + R^2 M_1}\right) + x_2\left(\frac{-RK_{s2}}{J_M + R^2 M_1}\right) = \frac{-R(M_1 g + F)}{J_M + R^2 M_1} \qquad (1)$$

$$\sum F_{M2} = K_{s2}(x_2 - \theta R) - M_2 g = -M_2 \ddot{x}_2$$

$$\ddot{x}_2 + x_2\left(\frac{K_{s2}}{M_2}\right) + \theta\left(\frac{-RK_{s2}}{M_2}\right) = g \qquad (2)$$

b)

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \theta\left(\frac{-K_{s1} - R^2 K_{s2}}{J_M + R^2 M_1}\right) + x_2\left(\frac{RK_{s2}}{J_M + R^2 M_1}\right) + \left(\frac{-RM_1 g - RF}{J_M + R^2 M_1}\right)$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_2 = \theta\left(\frac{RK_{s2}}{M_2}\right) + x_2\left(\frac{-K_{s2}}{M_2}\right) + g$$

5.

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \theta\left(\frac{-K_{s1} - r^2 K_{s2}}{J_M}\right) + \omega\left(\frac{-K_{d1}}{J_M}\right) + x\left(\frac{K_{s2} r}{J_M}\right) + \frac{Fr}{J_M}$$

$$\dot{x} = v$$

$$\dot{v} = \theta\left(\frac{K_{s2} r}{M}\right) + v\left(\frac{-K_{d2}}{M}\right) + x\left(\frac{-K_{s2} - K_{s3}}{M}\right)$$

6.

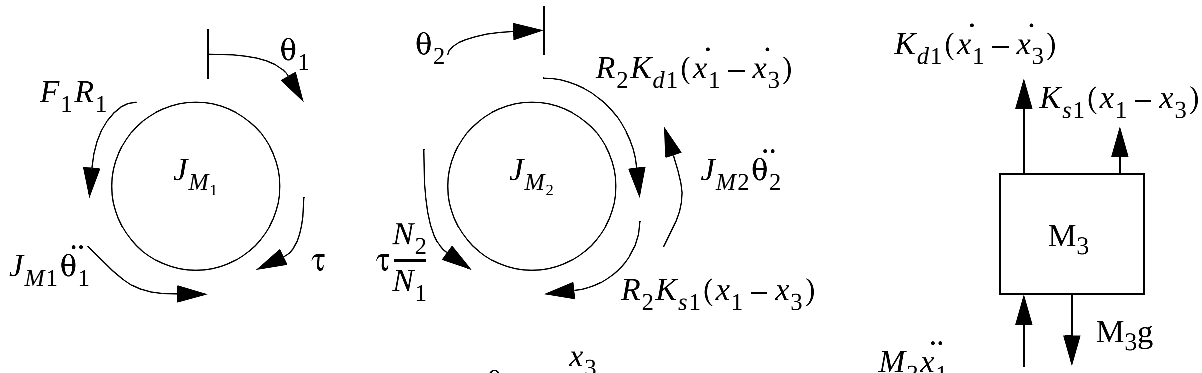
$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = v_1\left(\frac{-K_{d1}}{M_1}\right) + x_1\left(\frac{-K_{s1}}{M_1}\right) + \omega_1\left(\frac{K_{d1} R_2 R_4}{M_1 R_3}\right) + \theta_1\left(\frac{K_{s1} R_2 R_4}{M_1 R_3}\right) + g$$

$$\dot{\theta}_1 = \omega_1$$

$$\dot{\omega}_1 = v_1\left(\frac{K_{d1} R_2 R_4}{J_1 R_3}\right) + x_1\left(\frac{K_{s1} R_2 R_4}{J_1 R_3}\right) + \omega_1\left(\frac{-K_{d1} R_2^2 R_4^2}{J_1 R_3^2}\right) + \theta_1\left(\frac{-K_{s1} R_2^2 R_4^2}{J_1 R_3^2}\right) + \frac{-F_1 R_1}{J_1}$$

7.



$$\theta_1 N_1 = \theta_2 N_2 \qquad \theta_2 = \frac{x_3}{R_2}$$

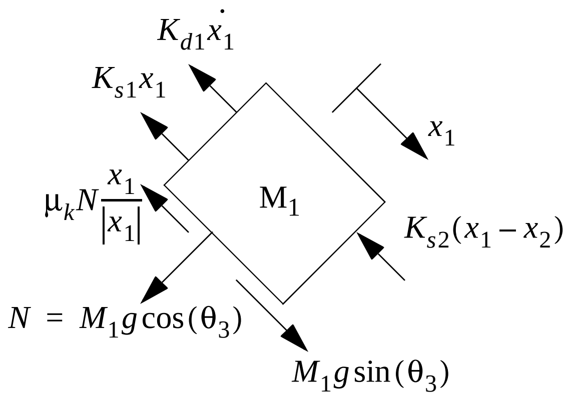
$$\dot{x}_1 = v_1$$

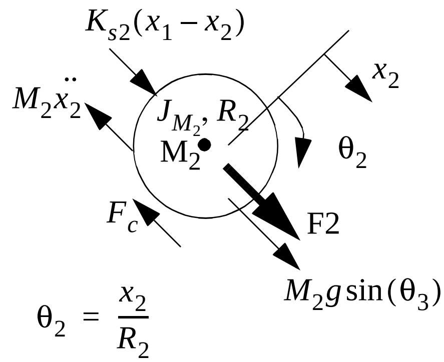
$$\dot{v}_1 = v_1\left(\frac{-K_{d1}}{M_3}\right) + x_1\left(\frac{-K_{s1}}{M_3}\right) + \omega_2\left(\frac{R_2K_{d1}}{M_3}\right) + \theta_1\left(\frac{R_2K_{s1}}{M_3}\right) - g$$

$$\dot{\theta}_2 = \omega_2$$

$$\dot{\omega}_2 = \frac{v_1(R_2K_{d1}) + x_1(R_2K_{s1}) + \omega_2(-R_2^2K_{d1}) + \theta_1(-R_2^2K_{s1}) + F\left(\frac{-N_2}{N_1}R_1\right)}{J_{M1}\frac{N_2^2}{N_1^2} + J_{M2}}$$

8.





$$\dot{x}_1 = v_1$$

$$\dot{v}_1 =$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_2 =$$

9.

$$\dot{x}_2 = v_2$$

$$\dot{v}_2 = x_3\left(\frac{R_1^2 R_2^2(-K_{s2} - K_{s3})}{4J_1R_2^2 + J_2R_1^2 + R_1^2R_2^2M_2}\right) + x_2\left(\frac{R_1^2R_2^2K_{s2}}{4J_1R_2^2 + J_2R_1^2 + R_1^2R_2^2M_2}\right) + \frac{R_1^2R_2^2(2F_1 + M_2 g)}{4J_1R_2^2 + J_2R_1^2 + R_1^2R_2^2M_2}$$

$$\dot{x}_3 = v_3$$

$$\dot{v}_3 = v_3\left(\frac{-K_{d3}}{M_3}\right) + x_3\left(\frac{-K_{s2} - K_{s3}}{M_3}\right) + x_2\left(\frac{K_{s2}}{M_3}\right) + g$$

10.

state equations

$$\dot{x}_1 = v_1$$
$$\dot{v}_1 = F_1\left(\frac{R_1 + R_2}{R_1 M_1}\right)$$
$$\dot{q} = x_2\left(\frac{-K_{s2}}{K_{d2}}\right) + x_3\left(\frac{K_{s2}}{K_{d2}}\right) + \frac{F_1}{K_{d1}}$$
$$\dot{p} = x_1\left(\frac{-K_{s1}}{K_{d1}}\right) + x_2\left(\frac{R_1 K_{s1}}{K_{d1}(R_1 + R_2)}\right) + F_1\left(\frac{R_1 + R_2}{R_1 K_{d1}}\right)$$

output equations

$$x_2 = (x_1 - p)\left(\frac{R_1 + R_2}{R_1}\right)$$
$$x_3 = -q + x_2$$

11.

$$\dot{x}_1 = v_1$$
$$\dot{v}_1 = x_1\left(\frac{-K_{s1}}{M_1}\right) + \theta_1\left(\frac{R_2 K_{s1} N_1}{M_1 N_2}\right) + g$$
$$\dot{\theta}_1 = \omega_1$$
$$\dot{\omega}_1 = x_1\left(\frac{R_2 K_{s1} N_1 N_2}{J_1 N_2^2 + J_2 N_1^2}\right) + \omega_1\left(\frac{-K_{d1} N_1^2}{J_1 N_2^2 + J_2 N_1^2}\right) + \theta_1\left(\frac{-R_2^2 K_{s1} N_1^2}{J_1 N_2^2 + J_2 N_1^2}\right) + \frac{F_1 R_1 N_2^2}{J_1 N_2^2 + J_2 N_1^2}$$

12.

For area: $$J_{area} = \int_0^R r^2 dA = \int_0^R r^2 (2\pi r dr) = 2\pi \int_0^R r^3 dr = 2\pi \left.\frac{r^4}{4}\right|_0^R = \frac{\pi R^4}{2}$$

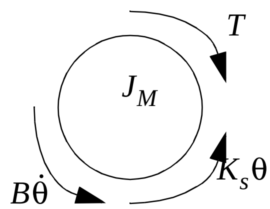
For mass: $$\rho = \frac{M}{A} = \frac{M}{\pi R^2}$$

$$J_{mass} = \int_0^R r^2 dM = \int_0^R r^2 (\rho 2\pi r dr) = 2\pi\rho \int_0^R r^3 dr = 2\pi\rho \left.\frac{r^4}{4}\right|_0^R = \rho\left(\frac{\pi R^4}{2}\right) = \frac{MR^2}{2}$$

The mass moment can be found by multiplying the area moment by the area density.

13.

a)



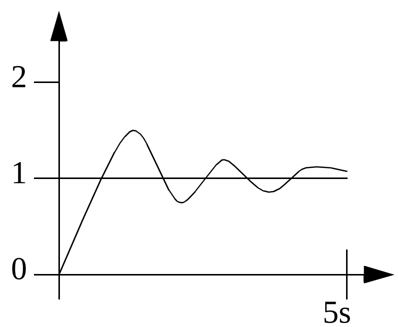
$$\overset{\curvearrowleft}{+}\ \sum M = T - K_s\theta - B\dot{\theta} = J_M\ddot{\theta}$$

$$J_M\ddot{\theta} + B\dot{\theta} + K_s\theta = T$$

$$\ddot{\theta} + \dot{\theta}\frac{B}{J_M} + \theta\frac{K_s}{J_M} = \frac{T}{J_M}$$

b) $\theta' = \omega$

$$\dot{\omega} = \frac{T}{J_M} - \frac{K_s}{J_M}\theta - \frac{B}{J_M}\omega$$



$$\dot{\omega} = \frac{10Nm}{1Kgm^2} - \frac{10\frac{Nm}{rad}}{1Kgm^2}\theta - \frac{1\frac{Nms}{rad}}{1Kgm^2}\omega$$

$$\dot{\omega} = \frac{Nm}{Kgm^2}\left(10 - \frac{10\theta}{rad} - \frac{s}{rad}\omega\right)$$

$$\dot{\omega} = \frac{\left(\frac{Kgm}{s^2}\right)m}{Kgm^2}\left(10 - \frac{10\theta}{rad} - \frac{s}{rad}\omega\right)$$

$$\dot{\omega} = s^{-2}\left(10 - \frac{10\theta}{rad} - \frac{s}{rad}\omega\right)$$

(c homogeneous:

$$\ddot{\theta} + (1s^{-1})\dot{\theta} + (10s^{-2})\theta = 0$$

guess: $\theta_h = e^{At}$ $\dot{\theta}_h = Ae^{At}$ $\ddot{\theta}_h = A^2e^{At}$

$$A^2e^{At} + (1s^{-1})Ae^{At} + (10s^{-2})e^{At} = 0$$

$$A^2 + (1s^{-1})A + 10s^{-2} = 0 \qquad A = \frac{-1s^{-1} \pm \sqrt{(1s^{-1})^2 - 4(1)(10s^{-2})}}{2(1)}$$

$$A = \frac{-1s^{-1} \pm \sqrt{1s^{-2} - 40s^{-2}}}{2(1)} = (-0.5 \pm j3.123)s^{-1}$$

$$\theta_h = C_1 e^{-0.5s^{-1}t}\cos(3.123s^{-1}t + C_2)$$

particular:

$$\theta'' + (1s^{-1})\theta' + (10s^{-2})\theta = 10s^{-2}$$

guess: $\theta_p = A$ $\dot{\theta}_p = 0$ $\ddot{\theta}_p = 0$

$$(0) + (1s^{-1})(0) + (10s^{-2})(A) = 10s^{-2} \qquad A = \frac{10s^{-2}}{10s^{-2}} = 1$$

$$\theta_p = 1$$

Initial conditions:

$$\theta(t) = C_1 e^{-0.5s^{-1}t}\cos(3.123s^{-1}t + C_2) + 1$$

$$\theta(0) = C_1 e^{-0.5s^{-1}0}\cos(3.123s^{-1}0 + C_2) + 1 = 0$$

$$C_1\cos(C_2) + 1 = 0$$

$$\theta'(t) = -0.5s^{-1}C_1e^{-0.5s^{-1}t}\cos(3.123s^{-1}t + C_2) - 3.123s^{-1}C_1e^{-0.5s^{-1}t}\sin(3.123s^{-1}t + C_2)$$

$$\theta'(0) = -0.5s^{-1}C_1(1)\cos(C_2) - 3.123s^{-1}C_1(1)\sin(C_2) = 0$$

$$-0.5\cos(C_2) - 3.123\sin(C_2) = 0$$

$$\frac{\sin(C_2)}{\cos(C_2)} = \frac{-0.5}{3.123} = \tan(C_2) \qquad C_2 = -0.159$$

$$C_1\cos(-0.159) + 1 = 0 \qquad C_1 = \frac{-1}{\cos(-0.159)} = -1.013$$

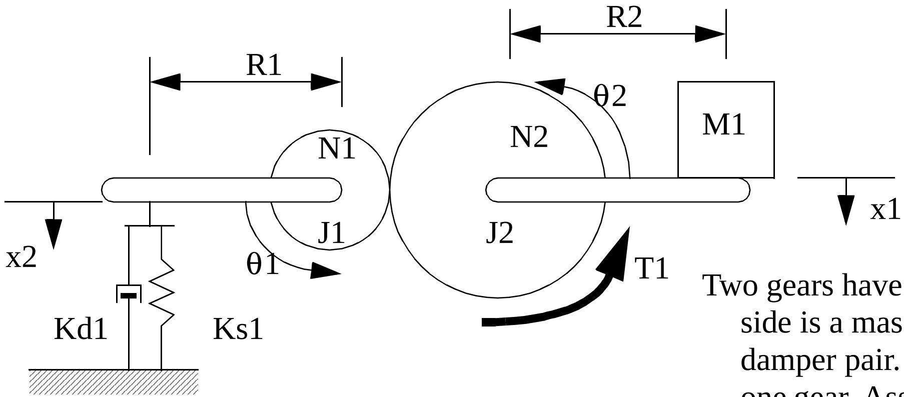
$$\theta(t) = -1.013e^{-0.5s^{-1}t}\cos(3.123s^{-1}t - 0.159) + 1$$

14.

$$\theta(t) = \frac{10}{9} + (-1.283)e^{-1.5t}\cos\left(\frac{\sqrt{27}}{2}t - 0.524\right)$$

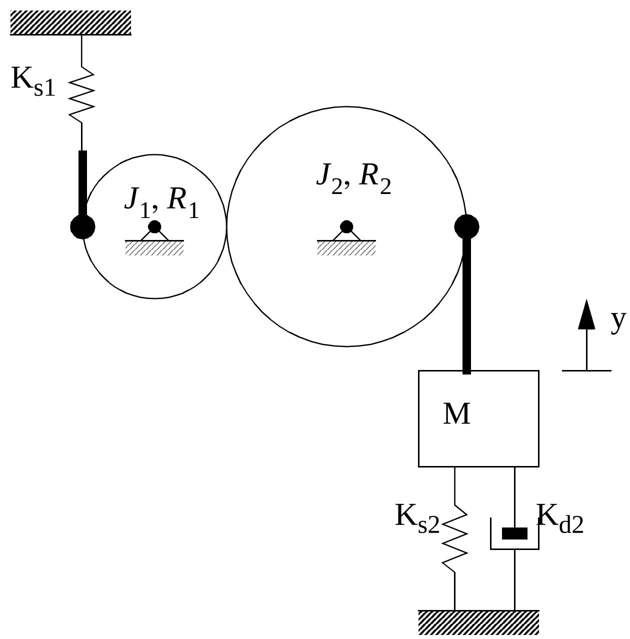
5.8 ASSIGNMENT PROBLEMS

1. for the system pictured below a) write the differential equations (assume small angular deflections) and b) put the equations in state variable form.

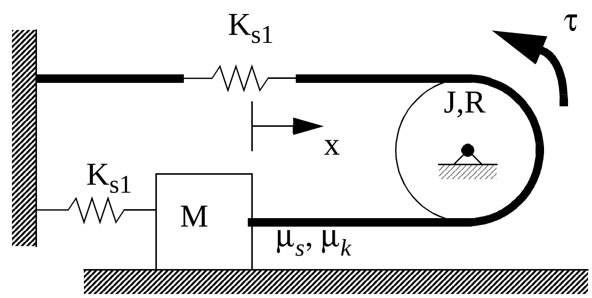


Two gears have levers attached. On one side is a mass, the other side a spring damper pair. A torque is applied to one gear. Assume the mass remains in contact with the lever.

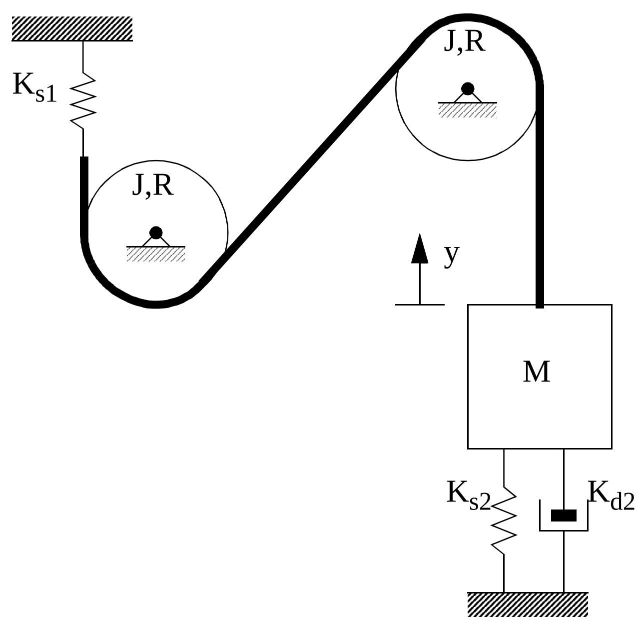
2. Draw FBDs for the following mechanical system containing two gears.



3. Draw FBDs for the following mechanical system. Consider both friction cases.

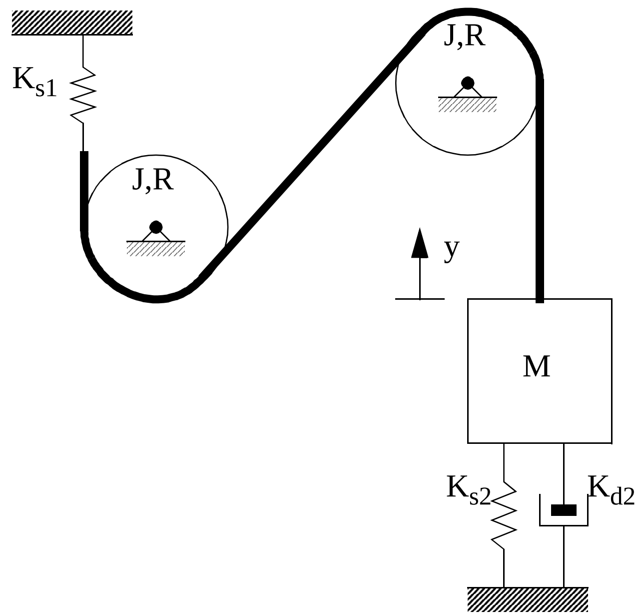


4. Draw FBDs for the following mechanical system.

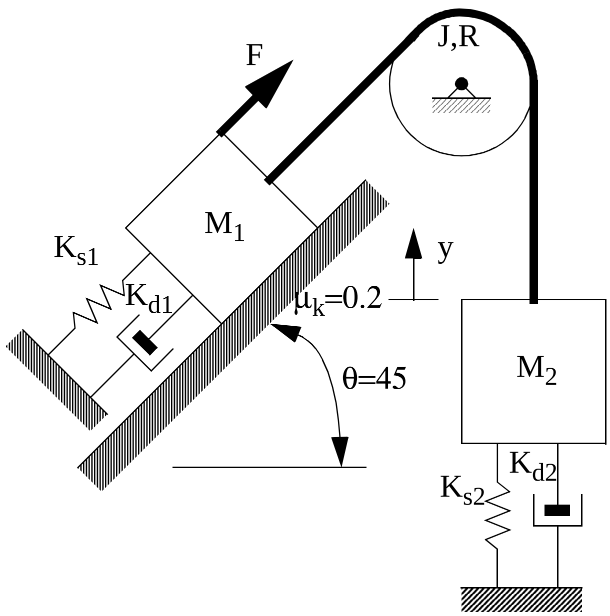


5. Develop a differential equation of motion for the system below assuming that the cable always

remains tight.



6. Analyze the system pictured below assuming the rope remains tight.



$$F = 10N$$

$$M_1 = 1kg \qquad K_{s1} = 100\frac{N}{m}$$

$$M_2 = 1kg \qquad K_{s2} = 100\frac{N}{m}$$

$$R = 0.1m \qquad K_{d1} = 50\frac{Ns}{m}$$

$$J = 10Kgm^2 \qquad K_{d2} = 50\frac{Ns}{m}$$

a) Draw FBDs and write the differential equations for the individual masses.
b) Combine the equations in input-output form with y as the output and F as the input.
d) Write the equations in state variable matrix form.
c) Use Runge-Kutta to find the system state after 1 second.

7. Analyze the system pictured below assuming the rope remains tight.

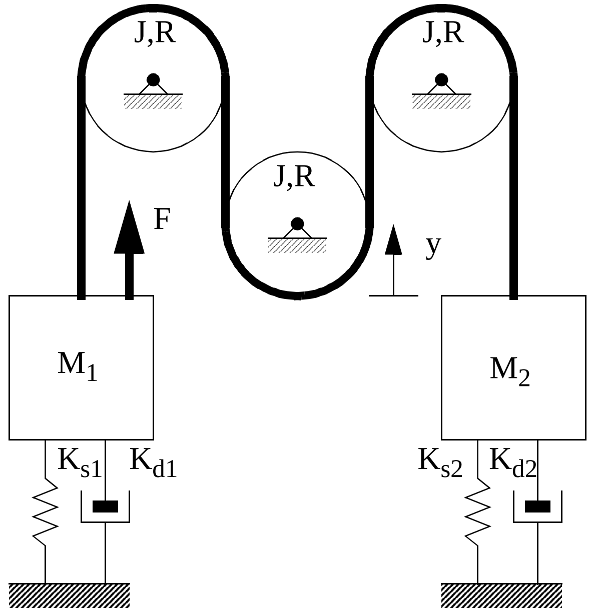
$$F = 10N$$

$$M_1 = 1kg \qquad K_{s1} = 100\frac{N}{m}$$

$$M_2 = 1kg \qquad K_{s2} = 100\frac{N}{m}$$

$$R = 0.1m \qquad K_{d1} = 50\frac{Ns}{m}$$

$$J = 10Kgm^2 \qquad K_{d2} = 50\frac{Ns}{m}$$



a) Draw FBDs and write the differential equations for the individual masses.
b) Combine the equations in input-output form with y as the output and F as the input.
c) Write the equations in state variable matrix form.
d) Use Runge-Kutta to find the system state after 1 second.