22. SYSTEM IDENTIFICATION

Topics:

Objectives:

22.1 INTRODUCTION

• A simple example

$$\frac{d}{dt}X_{out} = AX_{out} + BX_{in}$$

$$\frac{d}{dt}Y_{out} = CX_{out} + DX_{in}$$

Given a system model for a force applied to a mass,

$$\frac{d}{dt}x = v$$

$$\frac{d}{dt}v = \frac{F}{M}$$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{M} \end{bmatrix}$$

For an unknown second order linear system,

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}F$$

$$\frac{d}{dt}\begin{bmatrix} x_i \\ v_i \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_i \\ v_i \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}F$$

The state variables can be replaced with backwards difference equations,

$$\begin{bmatrix} \frac{x_i - x_{i-2}}{T} \\ \frac{x_i - 2x_{i-1} + x_{i-2}}{T^2} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_{i-1} \\ \frac{x_i - x_{i-2}}{T} \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}F$$

The equations can then be written for three different set of points to yield six equations.

$$\frac{x_i - x_{i-2}}{T} = ax_{i-1} + b\frac{x_i - x_{i-2}}{T} + eF$$

$$a(x_{i-1}) + b\left(\frac{x_i - x_{i-2}}{T}\right) + c(0) + d(0) + e(F) + f(0) = \frac{x_i - x_{i-2}}{T}$$

$$a(x_{i-4}) + b\left(\frac{x_{i-3} - x_{i-5}}{T}\right) + c(0) + d(0) + e(F) + f(0) = \frac{x_{i-3} - x_{i-5}}{T}$$

$$a(x_{i-7}) + b\left(\frac{x_{i-6} - x_{i-8}}{T}\right) + c(0) + d(0) + e(F) + f(0) = \frac{x_{i-6} - x_{i-8}}{T}$$

$$\frac{x_i - 2x_{i-1} + x_{i-2}}{T^2} = cx_{i-1} + d\frac{x_i - x_{i-2}}{T} + fF$$

$$a(0) + b(0) + c(x_{i-1}) + d\left(\frac{x_i - x_{i-2}}{T}\right) + e(0) + f(F) = \frac{x_i - 2x_{i-1} + x_{i-2}}{T^2}$$

$$a(0) + b(0) + c(x_{i-4}) + d\left(\frac{x_{i-3} - x_{i-5}}{T}\right) + e(0) + f(F) = \frac{x_{i-3} - 2x_{i-4} + x_{i-5}}{T^2}$$

$$a(0) + b(0) + c(x_{i-7}) + d\left(\frac{x_{i-6} - x_{i-7}}{T}\right) + e(0) + f(F) = \frac{x_{i-6} - 2x_{i-7} + x_{i-8}}{T^2}$$

The equations can be rewritten in matrix form,

$$\begin{bmatrix} x_{i-1} & \frac{x_i - x_{i-2}}{T} & 0 & 0 & F & 0 \\ x_{i-4} & \frac{x_{i-3} - x_{i-5}}{T} & 0 & 0 & F & 0 \\ x_{i-7} & \frac{x_{i-6} - x_{i-8}}{T} & 0 & 0 & F & 0 \\ 0 & 0 & x_{i-1} & \frac{x_i - x_{i-2}}{T} & 0 & F \\ 0 & 0 & x_{i-4} & \frac{x_{i-3} - x_{i-5}}{T} & 0 & F \\ 0 & 0 & x_{i-7} & \frac{x_{i-6} - x_{i-7}}{T} & 0 & F \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} \frac{x_i - x_{i-2}}{T} \\ \frac{x_{i-3} - x_{i-5}}{T} \\ \frac{x_{i-6} - x_{i-8}}{T} \\ \frac{x_i - 2x_{i-1} + x_{i-2}}{T^2} \\ \frac{x_{i-3} - 2x_{i-4} + x_{i-5}}{T^2} \\ \frac{x_{i-6} - 2x_{i-7} + x_{i-8}}{T^2} \end{bmatrix}$$

Some sample data points can be calculated for F=1 and M=1,

$$x = \frac{1}{2}at^2 = \frac{1}{2}\frac{F}{M}t^2 \qquad T = 1 \qquad F = 1 \qquad M = 1$$

$$x(0) = 0 \qquad x_{i-8} = 0$$

$$x(1) = 0.5 \qquad x_{i-7} = 0.5$$

$$x(2) = 2 \qquad x_{i-6} = 2$$

$$x(3) = 4.5 \qquad x_{i-5} = 4.5$$

$$x(4) = 8 \qquad x_{i-4} = 8$$

$$x(5) = 12.5 \qquad x_{i-3} = 12.5$$

$$x(6) = 18 \qquad x_{i-2} = 18$$

$$x(7) = 24.5 \qquad x_{i-1} = 24.5$$

$$x(8) = 32 \qquad x_i = 32$$

The values can be substituted into the matrix, and the matrix solved

$$\begin{bmatrix} 24.5 & \frac{32-18}{1} & 0 & 0 & 1 & 0 \\ 8 & \frac{12.5-4.5}{1} & 0 & 0 & 1 & 0 \\ 0.5 & \frac{2-0}{1} & 0 & 0 & 1 & 0 \\ 0 & 0 & 24.5 & \frac{32-18}{1} & 0 & 1 \\ 0 & 0 & 8 & \frac{12.5-4.5}{1} & 0 & 1 \\ 0 & 0 & 0.5 & \frac{2-0.5}{1} & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} \frac{32-18}{1} \\ \frac{12.5-4.5}{1} \\ \frac{2-0}{1} \\ \frac{32-2(24.5)+18}{1^2} \\ \frac{12.5-2(8)+4.5}{1^2} \\ \frac{2-2(0.5)+0}{1^2} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 24.5 & 14 & 0 & 0 & 1 & 0 \\ 8 & 8 & 0 & 0 & 1 & 0 \\ 0.5 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 24.5 & 14 & 0 & 1 \\ 0 & 0 & 8 & 8 & 0 & 1 \\ 0 & 0 & 0.5 & 1.5 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 14 \\ 8 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Substituting these values back into the state equation results in the following expression, which matches the exact state equation.

$$\frac{d}{dt}\begin{bmatrix} x_i \\ v_i \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_i \\ v_i \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}\begin{bmatrix} F \end{bmatrix} \qquad \text{state equation}$$

$$\begin{bmatrix} x_i \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_i \\ v_i \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} F \\ 1 \end{bmatrix} \qquad \text{output equation}$$

This can be incorporated as a realtime algorithm to calculate the data parameters based upon realtime updates.

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} x_{i-1} & \frac{x_i - x_{i-2}}{T} & 0 & 0 & F & 0 \\ x_{i-4} & \frac{x_{i-3} - x_{i-5}}{T} & 0 & 0 & F & 0 \\ x_{i-7} & \frac{x_{i-6} - x_{i-8}}{T} & 0 & 0 & F & 0 \\ 0 & 0 & x_{i-1} & \frac{x_i - x_{i-2}}{T} & 0 & F \\ 0 & 0 & x_{i-4} & \frac{x_{i-3} - x_{i-5}}{T} & 0 & F \\ 0 & 0 & x_{i-7} & \frac{x_{i-6} - x_{i-7}}{T} & 0 & F \end{bmatrix}^{-1} \begin{bmatrix} \frac{x_i - x_{i-2}}{T} \\ \frac{x_{i-3} - x_{i-5}}{T} \\ \frac{x_{i-6} - x_{i-8}}{T} \\ \frac{x_i - 2x_{i-1} + x_{i-2}}{T^2} \\ \frac{x_{i-3} - 2x_{i-4} + x_{i-5}}{T^2} \\ \frac{x_{i-6} - 2x_{i-7} + x_{i-8}}{T^2} \end{bmatrix}$$

This can then be used as a model to calculate a transfer function as shown below using the previous example.

The state equation variables are,

$$\frac{d}{dt}\begin{bmatrix} x_i \\ v_i \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_i \\ v_i \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}\begin{bmatrix} F \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{bmatrix} x_i \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_i \\ v_i \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} F \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

these can be used to calculate a SISO system transfer function, and its inverse.

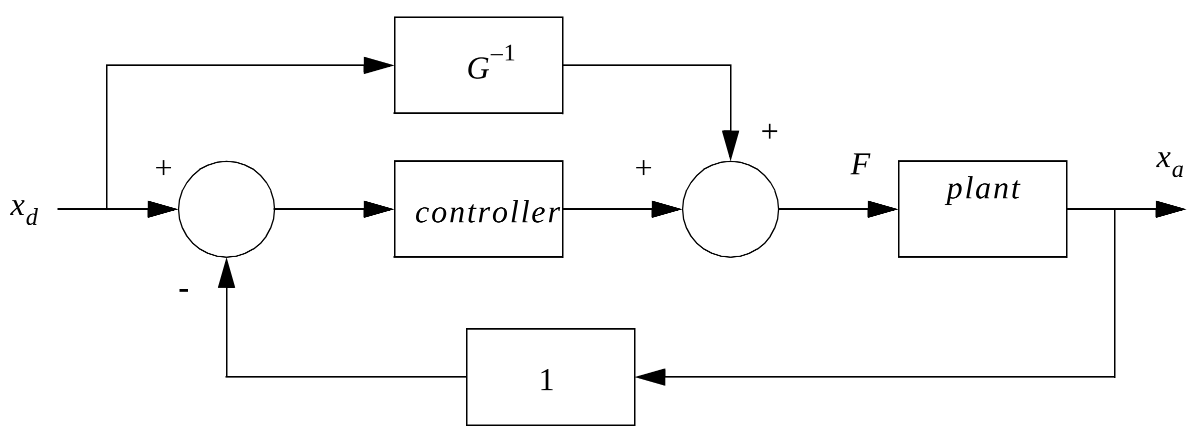
$$G = C(sI - A)^{-1}B + D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^{-1}\begin{bmatrix} e \\ f \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left(\begin{bmatrix} s-a & -b \\ -c & s-d \end{bmatrix}\right)^{-1}\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left(\frac{\begin{bmatrix} s-d & b \\ c & s-a \end{bmatrix}}{(s-a)(s-d)+bc}\right)\begin{bmatrix} e \\ f \end{bmatrix}$$

$$G = \left(\frac{\begin{bmatrix} s-d & b \\ 0 & 0 \end{bmatrix}}{(s-a)(s-d)+bc}\right)\begin{bmatrix} e \\ f \end{bmatrix} = \frac{(s-d)e + bf}{(s-a)(s-d)+bc}$$

$$G^{-1} = \frac{(s-a)(s-d)+bc}{(s-d)e + bf}$$

A block diagram for the system may then be written,



Consider the controller with the previous example.

The identified coefficients for the controller are substituted into the inverse transfer func.

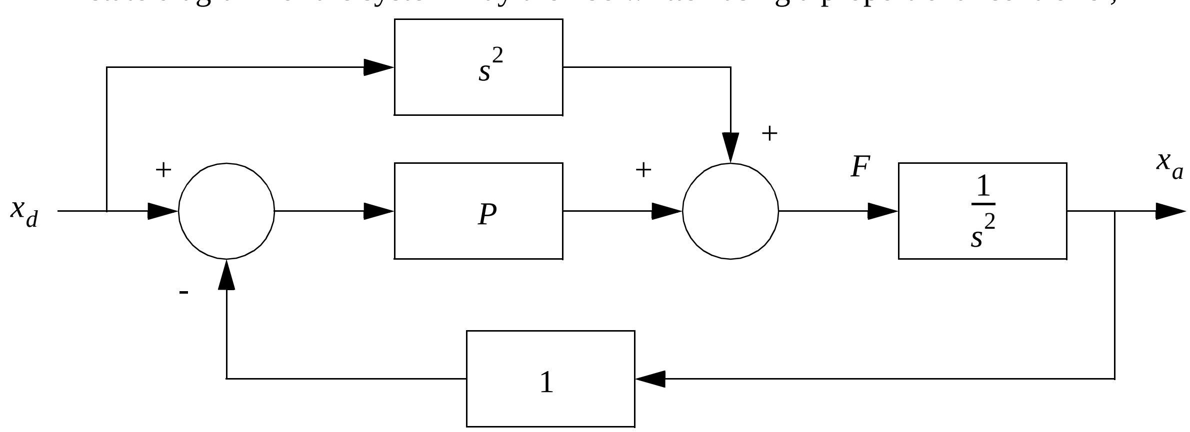
$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad G^{-1} = \frac{(s-a)(s-d)+bc}{(s-d)e+bf} = \frac{(s-0)(s-0)+1(0)}{(s-0)0+1(1)} = s^2$$

The identified coefficients for the controller are substituted into the inverse transfer func.

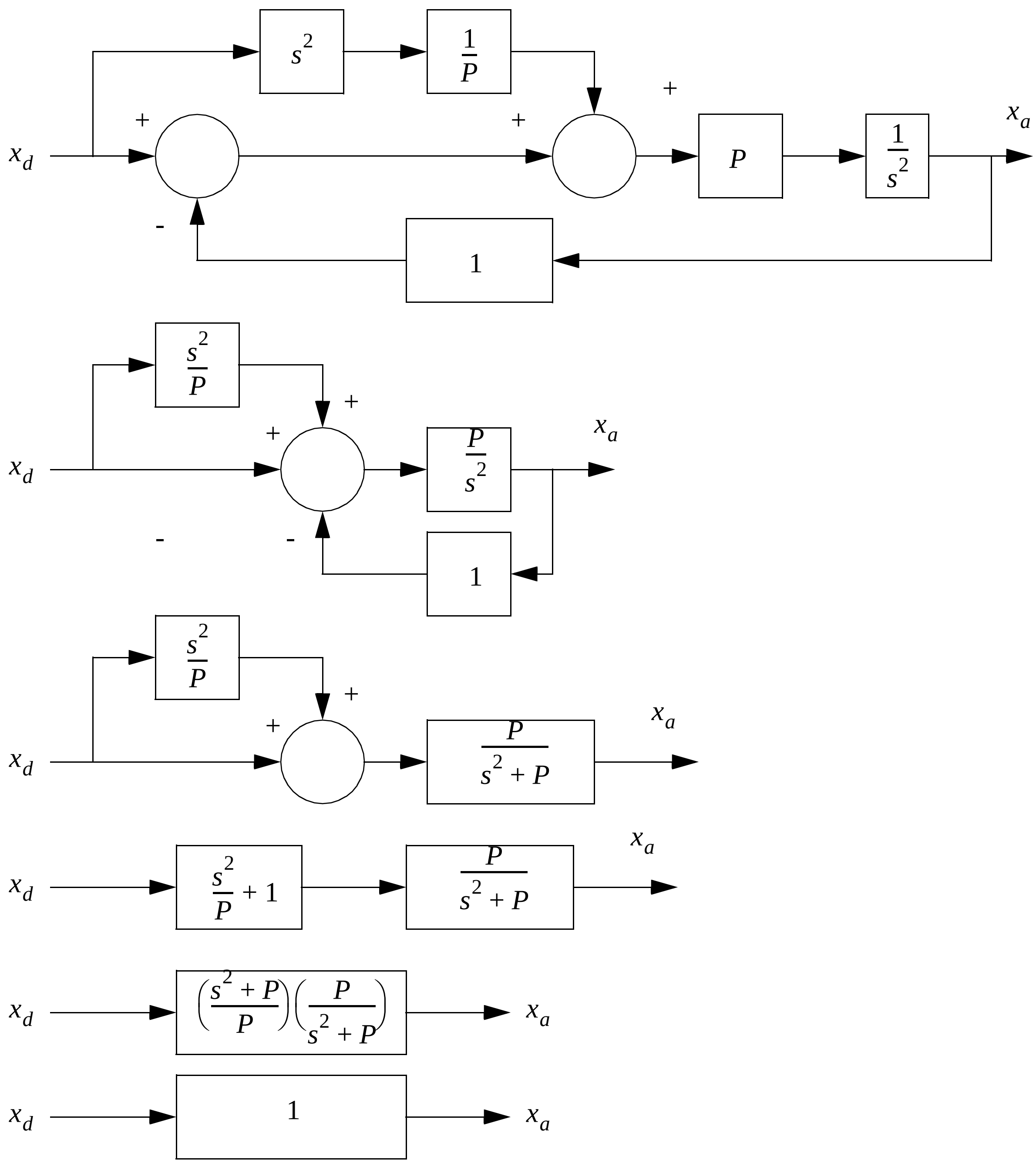
$$\left(\frac{d}{dt}\right)^2 x = \frac{F}{M} = \frac{F}{1}$$

$$\frac{x}{F} = \frac{1}{s^2}$$

A state diagram for the system may then be written using a proportional controller,



The state diagram is simplified,



Note: This suggests that the result is an ideal controller, however in practice there are errors, such as sensor noise, imperfect actuators, and changing system conditions such as mass. This also assumes that the force actuator can generate any requested force.

22.2 SUMMARY

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22.3 PRACTICE PROBLEMS

22.4 PRACTICE PROBLEM SOLUTIONS

22.5 ASSIGNMENT PROBLEMS

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