9. PHASOR ANALYSIS

Topics:
- Phasor forms for steady state analysis
- Complex and polar calculation of steady state system responses
- Vibration analysis

Objectives:
- To be able to analyze steady state responses using the phasor transform

9.1 INTRODUCTION

When a system is stimulated by an input it will respond. Initially there is a substantial transient response, that is eventually replaced by a steady state response. Techniques for finding the combined steady state and transient responses were covered in earlier chapters. These include the integration of differential equations, and numerical solutions. Phasor analysis can be used to find the steady state response only. These techniques involve using the phasor transform on the system transfer function, input and output.

9.2 PHASORS FOR STEADY-STATE ANALYSIS

When considering the differential operator we can think of it as a complex number, as in Figure 9.1. The real component of the number corresponds to the natural decay (e-to-the-t) of the system. But, the complex part corresponds to the oscillations of the system. In other words the real part of the number will represent the transient effects of the system, while the complex part will represent the sinusoidal steady-state. Therefore to do a steady-state sinusoidal analysis we can replace the 'D' operator with $j\omega$, this is the phasor transform.

$$D = \sigma + j\omega$$

where,

D = *differential operator*

σ = *decay constant*

ω = *oscillation frequency*

$$\boxed{\begin{array}{c}\textit{Phasor transform}\\ D = j\omega\end{array}}$$

Figure 9.1 Transient and steady-state parts of the differential operator

An example of the phasor transform is given in Figure 9.2. We start with a transfer function for a mass-spring-damper system. In this example numerical values are assumed to put the equation in a numerical form. The differential operator is replaced with $j\omega$, and the equation is simplified to a complex number in the denominator. This equation then described the overall response of the system to an input based upon the frequency of the input. A generic form of sinusoidal input for the system is defined, and also converted to phasor (complex) form. (Note: the frequency of the input does not show up in the complex form of the input, but it will be used later.) The steady state response of the system is then obtained by multiplying the transfer function by the input, to obtain the output.

A phasor transform can be applied to a transfer function for a mass-spring-damper system. Some component values are assumed.

$$M = 1000kg \qquad K_s = 2000\frac{N}{m} \qquad K_d = 3000\frac{Ns}{m}$$

$$\frac{x(D)}{F(D)} = \frac{1}{MD^2 + K_dD + K_s} = \frac{1}{1000D^2 + 3000D + 2000}$$

$$\frac{x(\omega)}{F(\omega)} = \frac{1}{1000j^2\omega^2 + 3000j\omega + 2000} = \frac{1}{(2000 - 1000\omega^2) + j(3000\omega)}$$

A given input function can also be converted to phasor form.

$$F(t) = A\sin(\omega_{input}t + \theta_{input})$$

$$F(\omega) = A(\cos\theta_{input} + j\sin\theta_{input})$$

$$F(\omega) = A\cos\theta_{input} + jA\sin\theta_{input}$$

Note: the frequency is not used when converting an oscillating signal to complex form. But it is needed for the transfer function.

The response of the steady state output 'x' can now be found for the given input.

$$x(\omega) = \frac{x(\omega)}{F(\omega)}F(\omega) = \frac{1}{(2000 - 1000\omega^2) + j(3000\omega)}(A\cos\theta_{input} + jA\sin\theta_{input})$$

Figure 9.2 A phasor transform example

To continue the example in Figure 9.2 values for the sinusoidal input force are assumed. After this the method only requires the simplification of the complex expression. In particular having a complex denominator makes analysis difficult and is undesirable. To simplify this expression it is multiplied by the complex conjugate. After this, the expression is quickly reduced to a simple complex number. The complex number is then converted to polar form, and then finally back into a function of time.

Assume the input to the system is, $F(t) = 10\sin(100t + 0.5)N$

$$A = 10N \qquad \omega = 100\frac{rad}{s} \qquad \theta_{input} = 0.5rad$$

These can be applied to find the steady state output response,

$$x(\omega) = \frac{1}{(2000 - 1000(100)^2) + j(3000(100))}(10\cos 0.5 + j10\sin 0.5)$$

$$x(\omega) = \frac{1}{(-9998000) + j(300000)}(8.776 + j4.794)$$

$$x(\omega) = \frac{8.776 + j4.794}{-9998000 + j300000}\left(\frac{(-9998000 - j300000)}{(-9998000 - j300000)}\right)$$

Note: This is known as the complex conjugate;
1. The value is equivalent to 1 so it does not change the value of the expression
2. The complex component is now negative
3. Only the denominator is used top and bottom

$$x(\omega) = \frac{(-86304248) + j(-50563212)}{1.0005\times10^{14}}$$

$$x(\omega) = (-0.863\times10^{-6}) + j(-0.505\times10^{-6})$$

$$x(\omega) = \sqrt{(-0.863\times10^{-6})^2 + (-0.505\times10^{-6})^2}\angle\operatorname{atan}\left(\frac{-0.505\times10^{-6}}{-0.863\times10^{-6}}\right) + \pi$$

Note: the signs of the components indicate that the angle is in the bottom left quadrant of the complex plane, so the angle should be between 180 and 270 degrees. To correct for this pi radians are added to the result of the calculation.

$$x(\omega) = 0.9999\times10^{-6}\angle 3.671$$

This can then be converted to a function of time.

$$x(t) = 0.9999\times10^{-6}\sin(100t + 3.671)m$$

Figure 9.3 A phasor transform example (cont'd)

Note: when dividing and multiplying complex numbers in polar form the magnitudes can be multiplied or divided, and the angles added or subtracted.

Unfortunately when the numbers are only added or subtracted they need to be converted back to cartesian form to perform the operations. This method eliminates the need to multiply by the complex conjugate.

$$\frac{A+jB}{C+jD} = \frac{\sqrt{A^2+B^2}\angle\operatorname{atan}\left(\frac{B}{A}\right)}{\sqrt{C^2+D^2}\angle\operatorname{atan}\left(\frac{D}{C}\right)} = \frac{\sqrt{A^2+B^2}}{\sqrt{C^2+D^2}}\angle\left(\operatorname{atan}\left(\frac{B}{A}\right)-\operatorname{atan}\left(\frac{D}{C}\right)\right)$$

$$\frac{A\angle\theta_1}{B\angle\theta_2} = \frac{A}{B}\angle(\theta_1-\theta_2)$$

$$(A\angle\theta_1)(B\angle\theta_2) = AB\angle(\theta_1+\theta_2)$$

For example,

$$(1+j)\left(\frac{2+j}{3+4j}\right)$$

$$= \sqrt{2}\angle 0.7854\frac{\sqrt{5}\angle 0.4636}{\sqrt{25}\angle 0.9273}$$

$$= \frac{\sqrt{2}\sqrt{5}}{\sqrt{25}}\angle 0.7854+0.4636-0.9273$$

$$= 0.6325\angle 0.3217$$

$$= 0.6+j0.2$$

Figure 9.4 Calculations in polar notation

The cartesian form of complex numbers seen in the last section are well suited to operations where complex numbers are added and subtracted. But, when complex numbers are to be multiplied and divided these become tedious and bulky. The polar form for complex numbers simplifies many calculations. The previous example started in Figure 9.2 is redone using polar notation in Figure 9.5. In this example the input is directly converted to polar form, without the need for calculation. The input frequency is substituted into the transfer function and it is then converted to polar form. After this the output is found by multiplying the transfer function by the input. The calculations for magnitudes involve simple multiplications. The angles are simply added. After this the polar form of the result is converted directly back to a function of time.

Consider the input function from the previous example in polar form it becomes,

$$F(t) = 10\sin(100t + 0.5)N \qquad F(\omega) = 10\angle 0.5$$

The transfer function can also be put in polar form.

$$\frac{x(\omega)}{F(\omega)} = \frac{1}{(2000 - 1000(100)^2) + j(3000(100))} = \frac{1}{-9998000 + j300000}$$

$$\frac{x(\omega)}{F(\omega)} = \frac{1\angle 0}{\sqrt{(-9998000)^2 + (300000)^2}\angle\left(\operatorname{atan}\left(\frac{300000}{-9998000}\right) + \pi\right)}$$

$$\frac{x(\omega)}{F(\omega)} = \frac{1\angle 0}{10002500\angle 3.112} = \frac{1}{10002500}\angle(0 - 3.112) = 0.9998\times 10^{-7}\angle -3.112$$

The output can now be calculated.

$$x(\omega) = \frac{x(\omega)}{F(\omega)}F(\omega) = (0.9998\times 10^{-7}\angle -3.112)(10\angle 0.5)$$

$$x(\omega) = 0.9998\times 10^{-7}(10)\angle(-3.112 + 0.5) = 0.9998\times 10^{-6}\angle -2.612$$

The output function can be written from this result.

$$x(\omega) = 0.9998\times 10^{-6}\sin(100t - 2.612)$$

Note: recall that $\tan\theta = \frac{Re}{Im}$ *but the* $\operatorname{atan}\theta$ *function in calculators and software only returns values between -90 to 90 degrees. To compensate for this the sign of the real and imaginary components must be considered to determine where the angle lies. If it lies beyond the -90 to 90 degree range the correct angle can be obtained by adding or subtracting 180 degrees.*

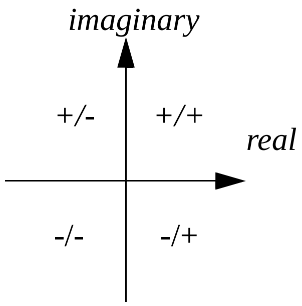
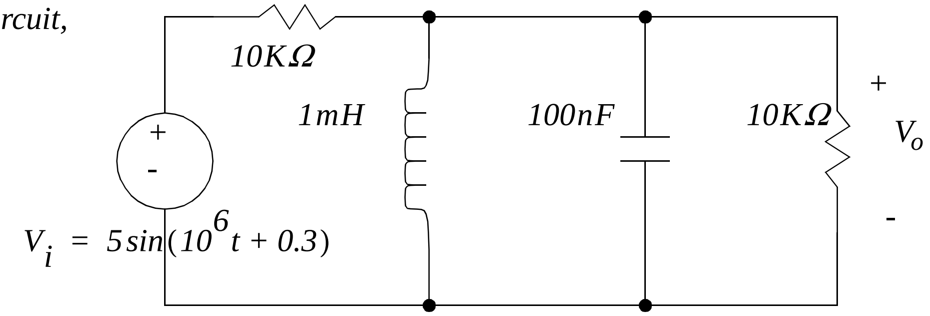


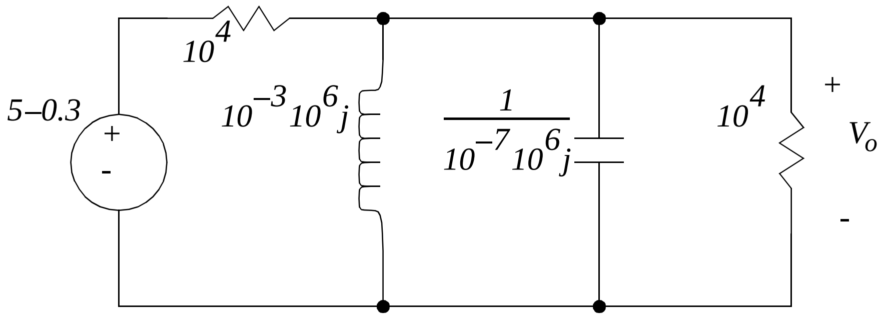
Figure 9.5 Correcting quadrants for calculated angles

Consider the circuit analysis example in Figure 9.6. In this example the component values are converted to their impedances, and the input voltage is converted to phasor form. (Note: this is a useful point to convert all magnitudes to powers of 10.) After this the three output impedances are combined to a single impedance. In this case the calculations were simpler in the cartesian form.

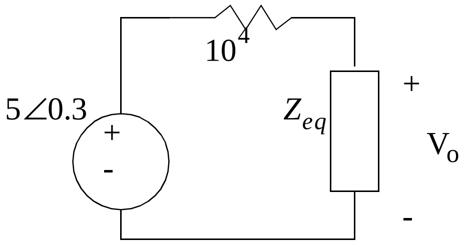
Given the circuit,



The impedances and input voltage can be written in phasor form.



The three output impedances in parallel can then be combined.



$$\frac{1}{Z_{eq}} = \frac{1}{10^{-3}10^6 j} + \frac{1}{\left(\frac{1}{10^{-7}10^6 j}\right)} + \frac{1}{10^4}$$

$$\frac{1}{Z_{eq}} = -10^{-3}j + 10^{-1}j + 10^{-4} = (10^{-4}) + j(0.099)$$

$$Z_{eq} = \frac{1}{(10^{-4}) + j(0.099)}$$

Figure 9.6 Phasor analysis of a circuit

The analysis continues in Figure 9.7 as the output is found using a voltage divider. In this case a combination of cartesian and polar forms are used to simplify the calculations. The final result is then converted back from phasor form to a function of time.

The output can be found using the voltage divider form.

$$V_o(\omega) = (5\angle 0.3)\left(\frac{Z_{eq}}{10^4 + Z_{eq}}\right)$$

$$V_o(\omega) = (5\angle 0.3)\left(\frac{\left(\frac{1}{(10^{-4}) + j(0.099)}\right)}{10^4 + \left(\frac{1}{(10^{-4}) + j(0.099)}\right)}\right)$$

$$V_o(\omega) = (5\angle 0.3)\left(\frac{1}{(10^4)((10^{-4}) + j(0.099)) + 1}\right)$$

$$V_o(\omega) = (5\angle 0.3)\left(\frac{1}{2 + j990}\right) = \frac{5\angle 0.3}{\sqrt{2^2 + 990^2}\angle \operatorname{atan}\frac{990}{2}} = \frac{5\angle 0.3}{990\angle 1.5687761}$$

$$V_o(\omega) = \frac{5}{990}\angle(0.3 - 1.5687761) = 5.05 \times 10^{-3}\angle -1.269$$

Finally, the output voltage can be written.

$$V_o(t) = 5.05\sin(10^6 t - 1.269)mV$$

Figure 9.7 Phasor analysis of a circuit (cont'd)

Phasor analysis is applicable to systems that are linear. This means that the principle of superposition applies. Therefore, if an input signal has more than one frequency component then the system can be analyzed for each component, and then the results simply added. The example considered in Figure 9.2 is extended in Figure 9.8. In this example the input has a static component, as well as frequencies at 0.5 and 20 rad/s. The transfer function is analyzed for each of these frequencies components. The output components are found by multiplying the inputs by the response at the corresponding frequency. The results are then converted back to functions of time, and added together.

Given the transfer function for the system,

$$\frac{x(\omega)}{F(\omega)} = \frac{1}{(2000 - 1000\omega^2) + j(3000\omega)}$$

and an input with multiple frequency components,

$$F(t) = 1000 + 20\sin(20t) + 10\sin((0.5t)(N))$$

the transfer function for each frequency can be calculated,

Note: these are gains and phase shifts that will be used heavily in Bode plots later.

$$\frac{x(0)}{F(0)} = \frac{1}{(2000 - 1000(0)^2) + j(3000(0))} = \frac{1}{2000} = 0.0005\angle 0$$

$$\frac{x(20)}{F(20)} = \frac{1}{(2000 - 1000(20)^2) + j(3000(20))} = \frac{1}{-398000 + j60000} = \frac{1\angle 0}{402497\angle 2.992}$$

$$\frac{x(0.5)}{F(0.5)} = \frac{1}{(2000 - 1000(0.5)^2) + j(3000(0.5))} = \frac{1}{1750 + j1500} = \frac{1\angle 0}{2305\angle 0.709}$$

These can then be multiplied by the input components to find output components.

$$x(0) = (0.0005\angle 0)1000\angle 0 = 0.5\angle 0$$

$$x(20) = \left(\frac{1\angle 0}{402497\angle 2.992}\right)20\angle 0 = 0.497\times 10^{-4}\angle -2.992$$

$$x(0.5) = \left(\frac{1\angle 0}{2305\angle 0.709}\right)10\angle 0 = 4.34 \times 10^{-3}\angle -0.709$$

Therefore the output is,

$$x(t) = 0.5 + 49.7 \times 10^{-6}\sin(20t - 2.992) + 4.34 \times 10^{-3}\sin(0.5t - 0.709)$$

Figure 9.8 A example for a signal with multiple frequency components (based on the example in Figure 9.2)

9.3 VIBRATIONS

Oscillating displacements and forces in mechanical systems will cause vibrations. In some cases these become a nuisance, or possibly lead to premature wear and failure in mechanisms. A common approach to dealing with these problems is to design vibration isolators. The equations for transmissibility and isolation is shown in Figure 9.9. These equations can compare the ratio of forces or displacements through an isolator. The calculation is easy to perform with a transfer function or Bode plot.

Given a vibration force in, to a force out,

$$T = \frac{F_{out}(\omega)}{F_{in}(\omega)} = \frac{x_{out}(\omega)}{x_{in}(\omega)}$$

The gain of a transfer function gives transmissibility

$$\%I = (1 - T)100\%$$

Figure 9.9 Transmissibility

Given the transfer function for a vibration isolator below, find a value of K that will give 50% isolation for a 10Hz vibration.

$$\frac{x_{out}}{x_{in}} = \frac{5D + 10}{4D^2 + 20KD + 4}$$

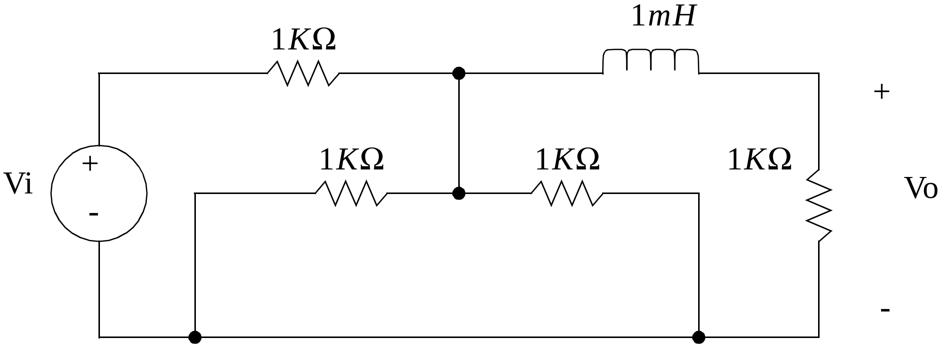
Figure 9.10 Drill problem: Select a K value

9.4 SUMMARY

- Phasor transforms and phasor representations can be used to find the steady state response of a system to a given input.
- Vibration analysis determines frequency components in mechanical systems.

9.5 PRACTICE PROBLEMS

1. Develop a transfer function for the system pictured below and then find the response to an input voltage of Vi = 10sin(1,000,000 t) using phasor transforms.

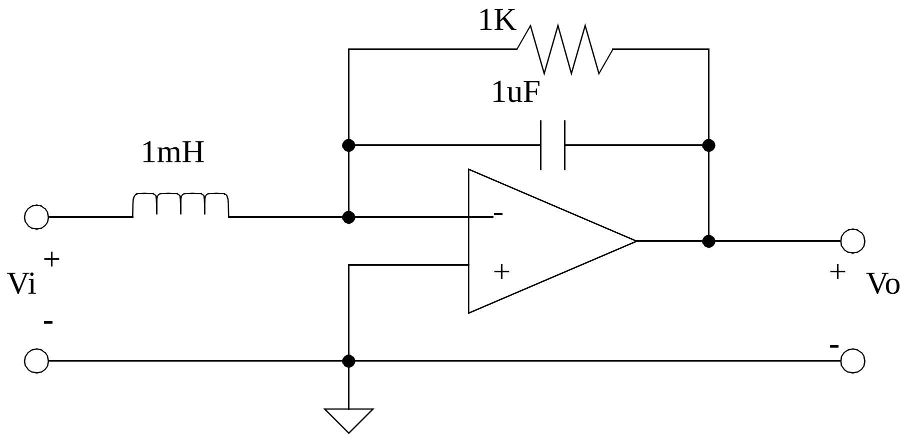


2. A single d.o.f. model with a weight of 1.2 kN and a stiffness of 340 N/m has a steady-state harmonic excitation force applied at 95 rpm (revolutions per minute). What damper value will give a vibration isolation of 92%?

3. Four helical compression springs are used, one at each corner of a piece of equipment. The spring rate is 240 N/m for each spring and the vertical static deflection of the equipment is 10mm. Calculate the mass of the equipment and determine the amount of isolation the springs would afford if the equipment operating frequency is twice the natural frequency of the system.

4. a) For the circuit below find the transfer function and the steady state response for an input of

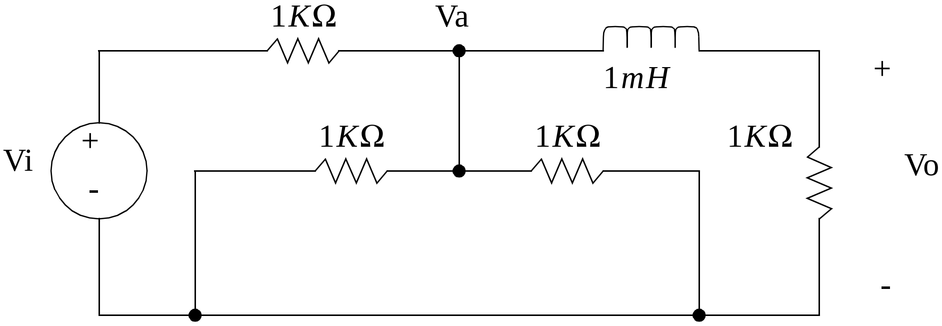
Vi=5sin(1000t)V.



b) Verify the results in part a) by explicitly solving the differential equation.

9.6 PRACTICE PROBLEM SOLUTIONS

1.



$$\sum I_{Va} = \frac{V_a - V_i}{1K\Omega} + \frac{V_a}{1K\Omega} + \frac{V_a}{1K\Omega} + \frac{V_a - V_o}{0.001D} = 0$$

$$V_a\left(\frac{3}{1K\Omega} + \frac{1}{0.001D}\right) + V_i\left(\frac{-1}{1K\Omega}\right) = V_o\left(\frac{1}{0.001D}\right) \qquad (1)$$

$$\sum I_{Vo} = \frac{V_o - V_a}{0.001D} + \frac{V_o}{1K\Omega} = 0$$

$$V_o\left(\frac{1}{0.001D} + \frac{1}{1K\Omega}\right) = V_a\left(\frac{1}{0.001D}\right)$$

$$V_o\left(\frac{0.001D + 1K\Omega}{1K\Omega}\right) = V_a \qquad (2)$$

substitute (2) into (1)

$$V_o\left(\frac{0.001D + 1K\Omega}{1K\Omega}\right)\left(\frac{3}{1K\Omega} + \frac{1}{0.001D}\right) + V_i\left(\frac{-1}{1K\Omega}\right) = V_o\left(\frac{1}{0.001D}\right)$$

$$\frac{V_o}{V_i} = \frac{\frac{1}{1K\Omega}}{\left(\frac{0.001D + 1K\Omega}{1K\Omega}\right)\left(\frac{3}{1K\Omega} + \frac{1}{0.001D}\right) - \left(\frac{1}{0.001D}\right)}$$

for the given input of Vi(t) = 10sin(1,000,000 t).

$$\frac{V_o}{10+0j} = \frac{\frac{1}{1K\Omega}}{\left(\frac{0.001j10^6+1K\Omega}{1K\Omega}\right)\left(\frac{3}{1K\Omega}+\frac{1}{0.001(j10^6)}\right)-\left(\frac{1}{0.001(j10^6)}\right)}$$

$$\frac{V_o}{10+0j} = \frac{10^{-3}}{(j+1)(3\times 10^{-3}-j10^{-3})+j(10^{-3})}$$

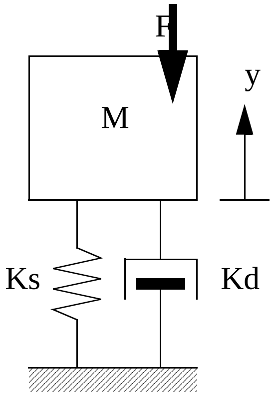
$$\frac{V_o}{10+0j} = \frac{1}{(j+1)(3-j)+j}$$

$$\frac{V_o}{10+0j} = \left(\frac{1}{4+3j}\right)\left(\frac{4-3j}{4-3j}\right)$$

$$V_o = 10\left(\frac{4-3j}{25}\right) = 2\angle -0.644$$

$$V_o(t) = 2.00\sin(10^6 t-0.644)V$$

2.



$$\sum F_y = -K_s y - K_d y D - F = M y D^2$$

$$\frac{y}{F} = \frac{-1}{D^2M+DK_d+K_s}$$

$$F_{floor} = K_s y + K_d y D$$

$$\frac{F_{floor}}{y} = K_s + K_d D$$

$$\frac{F_{floor}}{F} = \left(\frac{F_{floor}}{y}\right)\left(\frac{y}{F}\right) = \frac{-(K_s+K_dD)}{D^2M+DK_d+K_s}$$

$$\frac{F_{floor}}{F} = \frac{-(K_s+K_d\omega j)}{-\omega^2M+\omega jK_d+K_s}$$

$$\left|\frac{F_{floor}}{F}\right| = \frac{\sqrt{K_s^2+(K_d\omega)^2}}{\sqrt{(K_s-\omega^2M)^2+(\omega K_d)^2}}$$

(cont'd

For 92% isolation, there is 100-92 = 8% transmission, at 95rpm.

$$K_s = 340\frac{N}{m} \qquad M = \frac{1200N}{9.81\frac{N}{kg}} = 122kg$$

$$\omega = \left(95\frac{rev}{min}\right)\left(\frac{1min}{60\sec}\right)\left(\frac{2\pi rad}{rev}\right) = 9.95\frac{rad}{s}$$

$$0.08 = \frac{\sqrt{\left(340\frac{N}{m}\right)^2 + \left(K_d 9.95\frac{rad}{s}\right)^2}}{\sqrt{\left(340\frac{N}{m} - \left(9.95\frac{rad}{s}\right)^2 122kg\right)^2 + \left(9.95\frac{rad}{s}K_d\right)^2}}$$

$$\left(340\frac{N}{m} - \left(9.95\frac{rad}{s}\right)^2 122kg\right)^2 + \left(9.95\frac{rad}{s}K_d\right)^2 = \frac{\left(340\frac{N}{m}\right)^2 + \left(K_d 9.95\frac{rad}{s}\right)^2}{0.08^2}$$

$$(1.377878\times10^8)\frac{N^2}{m^2} + 99.0025\frac{rad^2}{s^2}K_d^2 = \frac{115600}{0.0064}\frac{N^2}{m^2} + K_d^2\frac{99.0025}{0.0064}\frac{rad^2}{s^2}$$

$$\left(\frac{1.197253\times10^8}{15370.138}\right)\frac{N^2 s^2}{m^2 rad^2} = K_d^2 \qquad K_d = 88.3\frac{Ns}{m}$$

3.

a) $M = 0.979kg$

b) $I = 67\%$

4.

a) $V_o(t) = 3536\sin(1000t + 0.785)$

b) $V_o(t) = -5000 + 2500e^{-1000t} + 3536\sin\left(1000t + \frac{\pi}{4}\right)$

9.7 ASSIGNMENT PROBLEMS

1. For the following transfer function,

a) Draw the Bode plot on the attached semi-log graph paper.
b) Given an input of F=5sin(62.82t), find the output, x, using the Bode plot.
c) Given an input of F=5sin(62.82t), find the output, x, using phasors.

$$\frac{x}{F} = \frac{D^2}{(D + 200\pi)^2}$$

2. For the following transfer function,
a) Draw the Bode plot on the attached semi-log graph paper.
b) Given an input of F=5sin(62.82t), find the output, x, using the Bode plot.
c) Given an input of F=5sin(62.82t), find the output, x, using phasors.

$$\frac{x}{F} = \frac{D(D + 2\pi)}{(D + 200\pi)^2}$$

3. For the following transfer function,
a) Draw the Bode plot on the attached semi-log graph paper.
b) Given an input of F=5sin(62.82t), find the output, x, using the Bode plot.
c) Given an input of F=5sin(62.82t), find the output, x, using phasors.

$$\frac{x}{F} = \frac{D^2(D + 2\pi)}{(D + 200\pi)^2}$$