7. ELECTRICAL SYSTEMS

Topics:
- Basic components; resistors, power sources, capacitors, inductors and op-amps
- Device impedance
- Example circuits

Objectives:
- To apply analysis techniques to circuits

7.1 INTRODUCTION

A voltage is a pull or push acting on electrons. The voltage will produce a current when the electrons can flow through a conductor. The more freely the electrons can flow, the lower the resistance of a material. Most electrical components are used to control this flow.

7.2 MODELING

Kirchoff's voltage and current laws are shown in Figure 7.1. The node current law holds true because the current flow in and out of a node must total zero. If the sum of currents was not zero then electrons would be appearing and disappearing at that node, thus violating the law of conservation of matter. The loop voltage law states that the sum of all rises and drops around a loop must total zero.

$$\sum I_{node} = 0 \qquad \textit{node current}$$

$$\sum V_{loop} = 0 \qquad \textit{loop voltage}$$

Figure 7.1 Kirchoff's laws

The simplest form of circuit analysis is for DC circuits, typically only requiring algebraic manipulation. In AC circuit analysis we consider the steady-state response to a sinusoidal input. Finally the most complex is transient analysis, often requiring integration, or similar techniques.

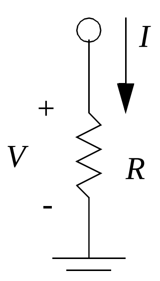
• DC (Direct Current) - find the response for a constant input.
• AC (Alternating Current) - find the steady-state response to an AC input.
• Transient - find the initial response to changes.

There is a wide range of components used in circuits. The simplest components are passive, such as resistors, capacitors and inductors. Active components are capable of changing their behaviors, such as op-amps and transistors. A list of components that will be discussed in this chapter are listed below.

• resistors - reduce current flow as described with ohm's law
• voltage/current sources - deliver power to a circuit
• capacitors - pass current based on current flow, these block DC currents
• inductors - resist changes in current flow, these block high frequencies
• op-amps - very high gain amplifiers useful in many forms

7.2.1 Resistors

Resistance is a natural phenomenon found in all materials except superconductors. A resistor will oppose current flow as described by ohm's law in Figure 7.2. The resistance value is assumed to be linear, but in actuality it varies with conductor temperature.

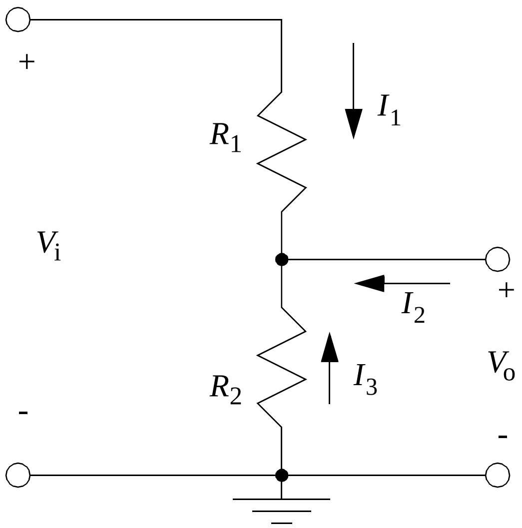


$$I = \frac{V}{R}$$

$$V = IR$$

Figure 7.2 Ohm's law

The voltage divider example in Figure 7.3 illustrates the methods for analysis of circuits using resistors. In this circuit an input voltage is supplied on the left hand side. The output voltage on the right hand side will be some fraction of the input voltage. If the output resistance is very large, no current will flow, and the ratio of output to input voltages is determined by the ratio of the resistance between R1 and R2. To prove this the currents into the center node are summed and set equal to zero. The equations are then manipulated to produce the final relationship.



$$I_1 + I_2 + I_3 = 0$$

$$\therefore \left(\frac{V_i - V_o}{R_1}\right) + I_2 + \left(\frac{0 - V_o}{R_2}\right) = 0$$

Assume the output resistance is large, so I_2 is negligible.

$$\therefore \left(\frac{V_i - V_o}{R_1}\right) + \left(\frac{-V_o}{R_2}\right) = 0$$

$$\therefore V_o\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = V_i\left(\frac{1}{R_1}\right)$$

$$\therefore V_o\left(\frac{R_1 + R_2}{R_1 R_2}\right) = V_i\left(\frac{1}{R_1}\right)$$

$$\boxed{\therefore \frac{V_o}{V_i} = \frac{R_2}{R_1 + R_2}}$$

Figure 7.3 A voltage divider circuit

If two resistors are in parallel or series they can be replaced with a single equivalent resistance, as shown in Figure 7.4.

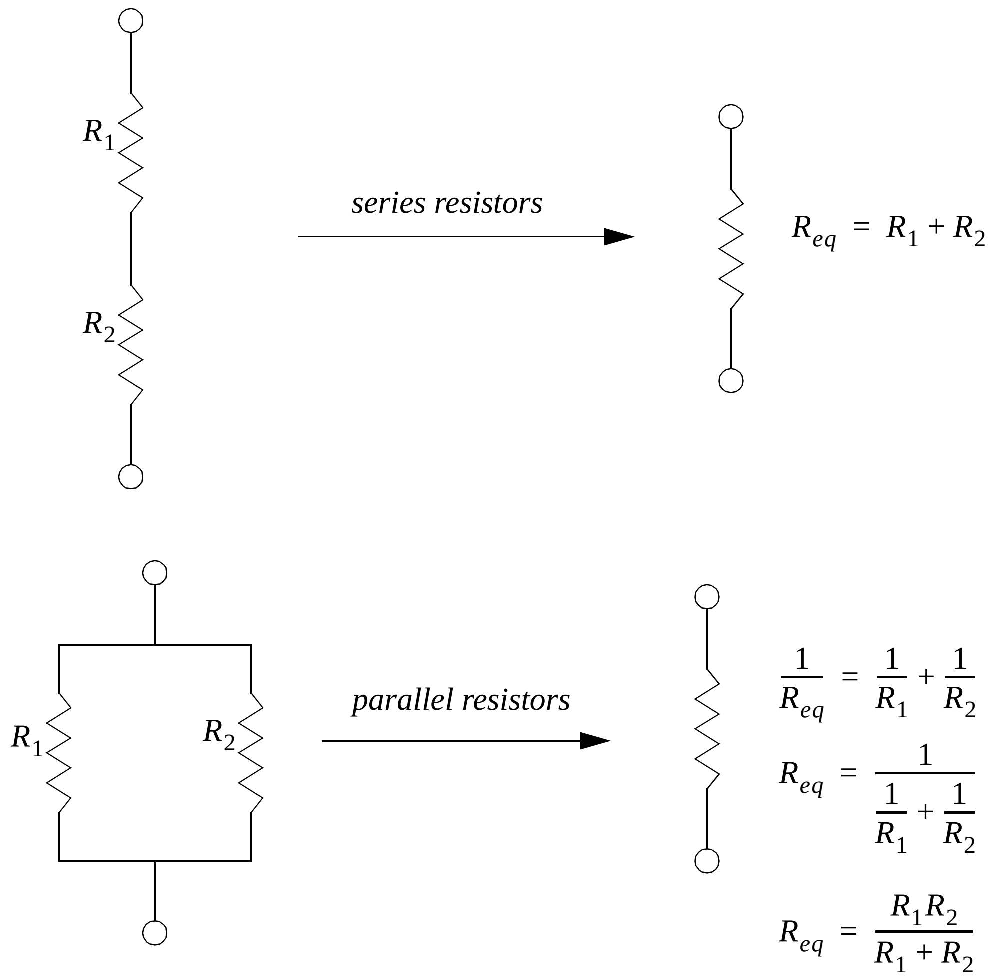


Figure 7.4 Equivalent resistances for resistors in parallel and series

7.2.2 Voltage and Current Sources

A voltage source will maintain a voltage in a circuit, by varying the current as required. A current source will supply a current to a circuit, by varying the voltage as required. The schematic symbols for voltage and current sources are shown in Figure 7.5. The supplies with '+' and '-' symbols provide DC voltages, with the symbols indicating polarity. The symbol with two horizontal lines is a battery. The circle with a sine wave is an AC voltage supply. The last symbol with an arrow inside the circle is a current supply. The arrow indicates the direction of positive current flow.

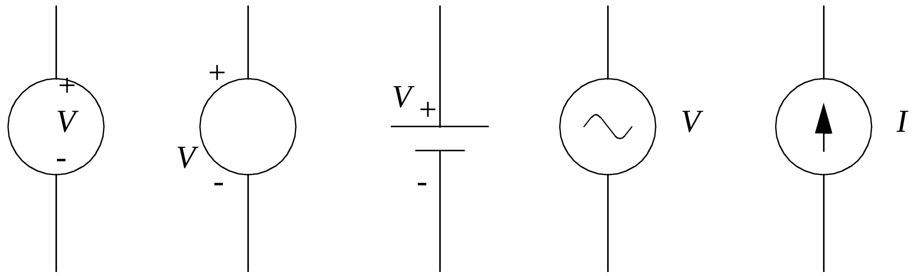
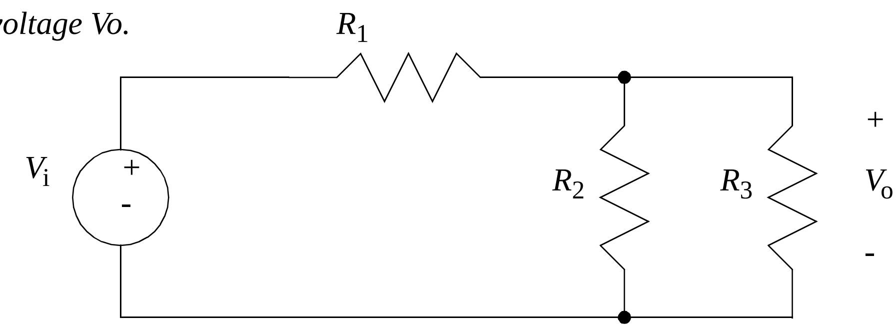


Figure 7.5 Voltage and current sources

A circuit containing a voltage source and resistors is shown in Figure 7.6. The circuit is analyzed using the node voltage method.

Find the output voltage Vo.



Examining the circuit there are two loops, but only one node, so the node current methods is the most suitable for calculations. The currents into the upper right node, Vo, will be solved.

$$\sum I = \frac{V_o - V_i}{R_1} + \frac{V_o}{R_2} + \frac{V_o}{R_3} = 0$$

$$V_o\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = V_i\left(\frac{1}{R1}\right)$$

$$V_o\left(\frac{R_2R_3 + R_1R_3 + R_1R_2}{R_1R_2R_3}\right) = V_i\left(\frac{1}{R1}\right)$$

$$V_o = V_i\left(\frac{R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}\right)$$

Aside: when doing node-current methods, select currents out of a node as positive, and in as negative. This will reduce the chances of careless mistakes.

Figure 7.6 A circuit calculation

Evaluate the circuit in Figure 7.6 using the loop voltage method.

$$V_o = V_i\left(\frac{R_2R_3}{R_1R_2 + R_1R_3 + R_2R_3}\right)$$

Figure 7.7 Drill problem: Mesh solution of voltage divider

Evaluate the voltage divider in Figure 7.3 using the loop current method. Hint: Put a voltage supply on the left, and an output resistor on the right. Remember that the output resistance should be infinite.

Figure 7.8 Drill problem: Mesh solution of voltage divider

Dependant (variable) current and voltage sources are shown in Figure 7.9. The voltage and current values of these supplies are determined by their relationship to some other circuit voltage or current. The dependant voltage source will be accompanied by a '+' and '-' symbol, while the current source has an arrow inside.

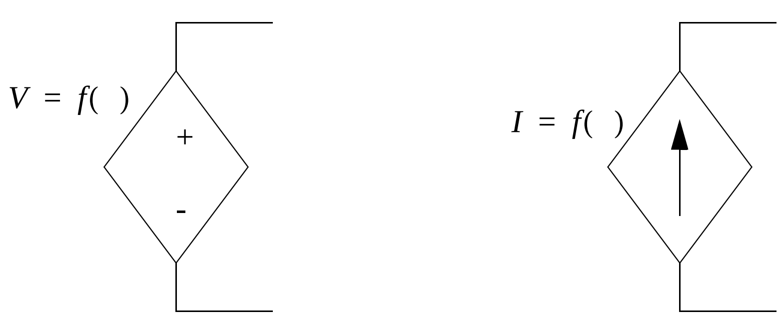
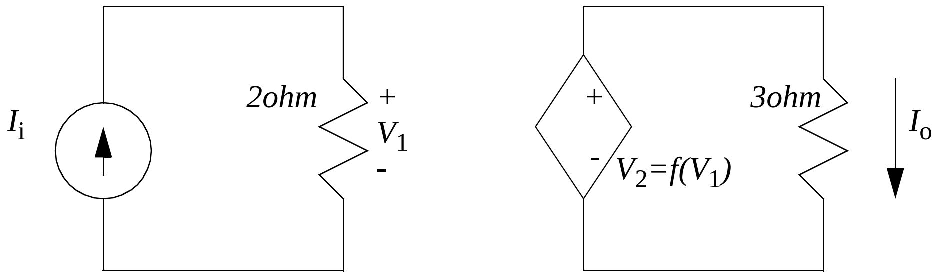


Figure 7.9 Dependant voltage sources



Find the output current, Io, if Ii = 1A.

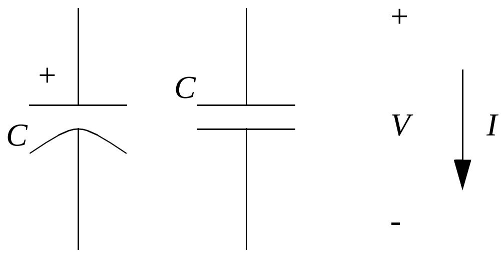
$f(V_1) = 3V_1$

What if the input current is Ii = 1sin(2t)A?

Figure 7.10 Drill problem: Find the currents in the circuit above

7.2.3 Capacitors

Capacitors are composed of two isolated metal plates very close together. When a voltage is applied across the capacitor, electrons will be forced into one plate, and forced out of the other plate. Temporarily this creates a small current flow until the plates reach equilibrium. So, any voltage change will result in some current flow. In practical terms this means that the capacitor will block any DC voltages, except for transient effects. But, high frequency AC currents will pass through the device. The equation for a capacitor and schematic symbols are given in Figure 7.11.



$$I = C\left(\frac{d}{dt}\right)V = CDV$$

Figure 7.11 Capacitors

The symbol on the left is for an electrolytic capacitor. These contain a special fluid that increases the effective capacitance of the device but requires that the positive and negative sides must be observed in the circuit. (Warning: reversing the polarity on an electrolytic capacitor can make them leak, fail and possibly explode.) The other capacitor symbol is for a regular capacitor, normally with values under a microfarad.

Find the current as a function of time.

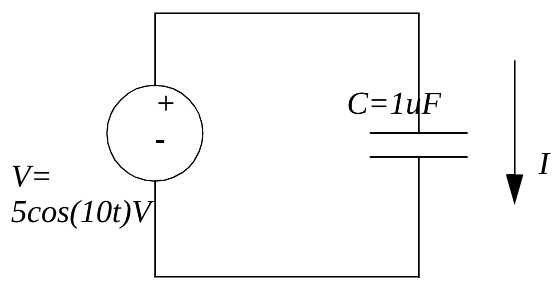
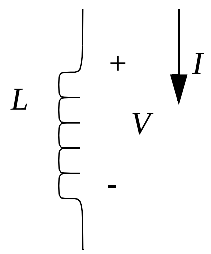


Figure 7.12 Drill problem: Current through a capacitor

7.2.4 Inductors

While a capacitor will block a DC current, an inductor will pass it. Inductors are basically coils of wire. When a current flows through the coils, a magnetic field is generated. If the current through the inductor changes then the magnetic field must change, otherwise the field is maintained without effort (i.e., no voltage). Therefore the inductor resists changes in the current. The schematic symbol and relationship for an inductor are shown in Figure 7.13.



$$V = L\left(\frac{d}{dt}\right)I = LDI$$

Figure 7.13 An inductor

An inductor is normally constructed by wrapping wire in loops about a core. The core can be hollow, or be made of ferrite to increase the inductance. Inductors usually cost more than capacitors. In addition, inductors are susceptible to interference when metals or other objects disturb their magnetic fields. When possible, designers normally try to avoid using inductors in circuits.

Find the current as a function of time.

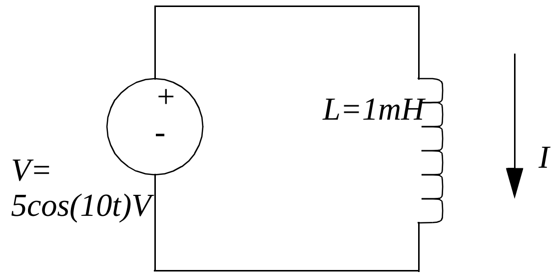


Figure 7.14 Drill problem: Current through an inductor

7.2.5 Op-Amps

The ideal model of an op-amp is shown in Figure 7.15. On the left hand side are the inverting and non-inverting inputs. Both of these inputs are assumed to have infinite impedance, and so no current will flow. Op-amp application circuits are designed so that the inverting and non-inverting inputs are driven to the same voltage level. The output of the op-amp is shown on the right. In circuits op-amps are used with feedback to perform standard operations such as those listed below.

- adders, subtractors, multipliers, and dividers - simple analog math operations
- amplifiers - increase the amplitude of a signal
- impedance isolators - hide the resistance of a circuit while passing a voltage

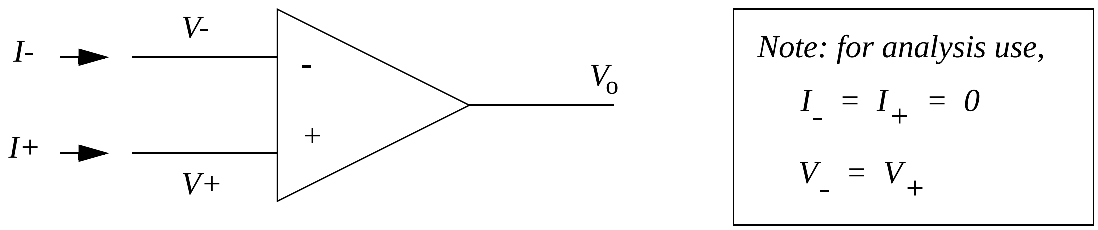
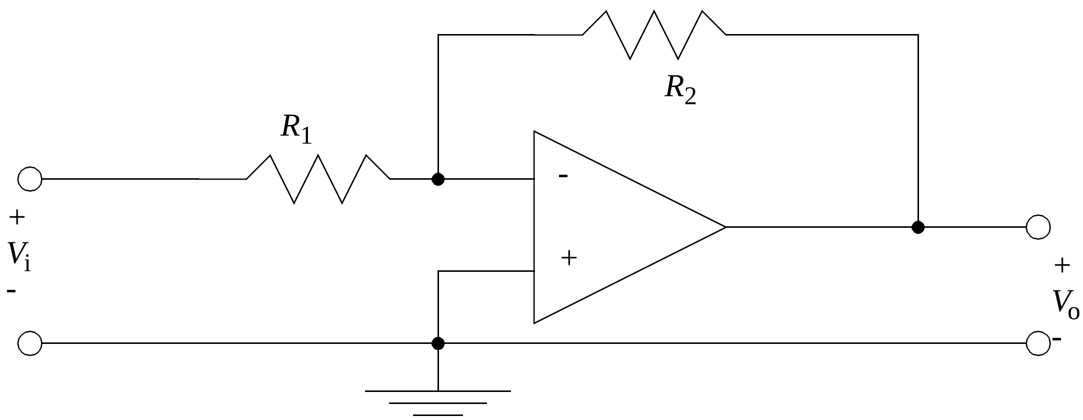


Figure 7.15 An ideal op-amp

A simple op-amp example is given in Figure 7.16. As expected both of the op-amp input voltages are the same. This is a function of the circuit design. (Note: most op-amp circuits are designed to force both inputs to have the same voltage, so it is normally reasonable to assume they are the same.) The non-inverting input is connected directly to ground, so it will force both of the inputs to 0V. When the currents are summed at the inverting input, an equation including the input and output voltages is obtained. The final equation shows the system is a simple multiplier, or amplifier. The gain of the amplifier is determined by the ratio of the input and feedback resistors.



The voltage at the non-inverting input will be 0V, by design the voltage at the inverting input will be the same.

$$V_+ = 0V$$

$$V_- = V_+ = 0V$$

The currents at the inverting input can be summed.

$$\sum I_{V-} = \frac{V_- - V_i}{R_1} + \frac{V_- - V_o}{R_2} = 0$$

$$\frac{0 - V_i}{R_1} + \frac{0 - V_o}{R_2} = 0$$

$$V_o = \frac{-R_2 V_i}{R_1}$$

$$V_o = \left(\frac{-R_2}{R_1}\right) V_i$$

Figure 7.16 A simple inverting operational amplifier configuration

An op-amp circuit that can subtract signals is shown in Figure 7.17.

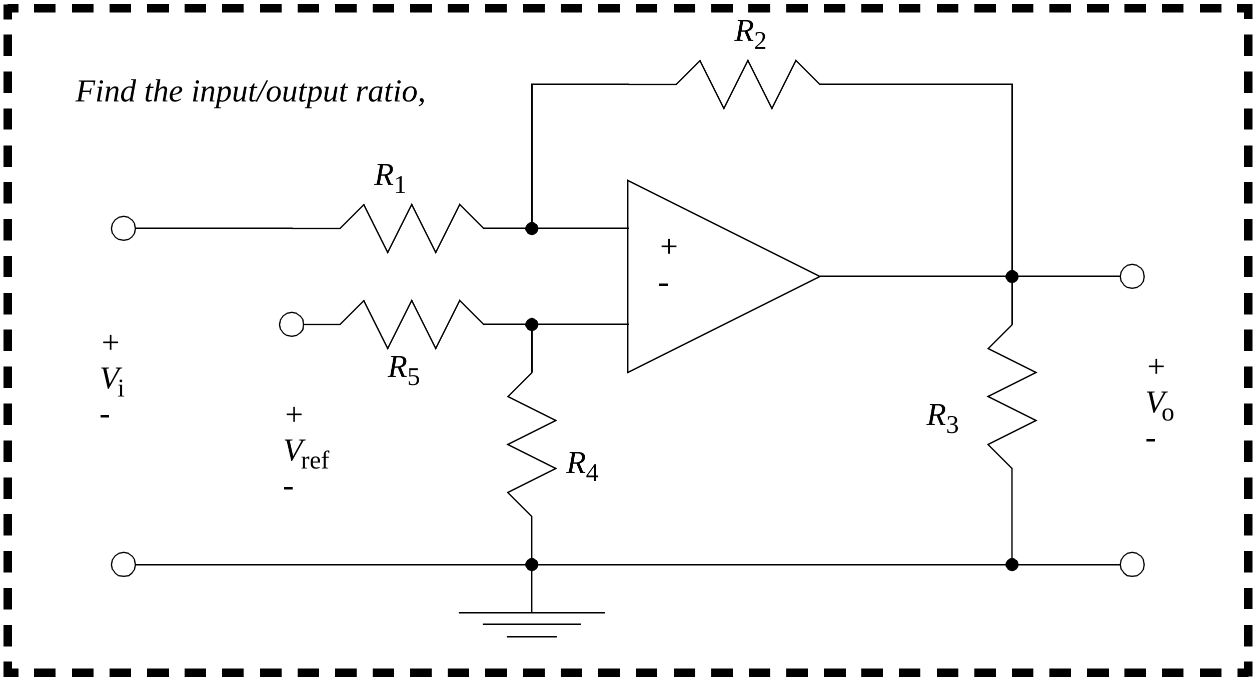


Figure 7.17 Op-amp example

For ideal op-amp problems the node voltage method is normally the best choice. The equations for the circuit in Figure 7.17 and derived in Figure 7.18. The general approach to this solution is to sum the currents into the inverting and non-inverting input nodes. Notice that the current into the op-amp is assumed to be zero. Both the inverting and non-inverting input voltages are then set to be equal. After that, algebraic manipulation results in a final expression for the op-amp. Notice that if all of the resistor values are the same then the circuit becomes a simple subtractor.

Note: normally node voltage methods work best with op-amp circuits, although others can be used if the non-ideal op-amp model is used.

First sum the currents at the inverting and non-inverting op-amp terminals.

$$\sum I_{V+} = \frac{V_+ - V_i}{R_1} + \frac{V_+ - V_o}{R_2} = 0$$

$$V_+\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = V_i\left(\frac{1}{R_1}\right) + V_o\left(\frac{1}{R_2}\right)$$

$$V_+\left(\frac{R_1 + R_2}{R_1 R_2}\right) = V_i\left(\frac{1}{R_1}\right) + V_o\left(\frac{1}{R_2}\right)$$

$$V_+ = V_i\left(\frac{R_2}{R_1 + R_2}\right) + V_o\left(\frac{R_1}{R_1 + R_2}\right) \qquad (1)$$

$$\sum I_{V-} = \frac{V_- - V_{ref}}{R_5} + \frac{V_-}{R_4} = 0$$

$$V_-\left(\frac{1}{R_4} + \frac{1}{R_5}\right) = V_{ref}\left(\frac{1}{R_5}\right)$$

$$V_- = V_{ref}\left(\frac{R_4}{R_4 + R_5}\right) \qquad (2)$$

Now the equations can be combined.

$$V_- = V_+$$

$$V_{ref}\left(\frac{R_4}{R_4 + R_5}\right) = V_i\left(\frac{R_2}{R_1 + R_2}\right) + V_o\left(\frac{R_1}{R_1 + R_2}\right)$$

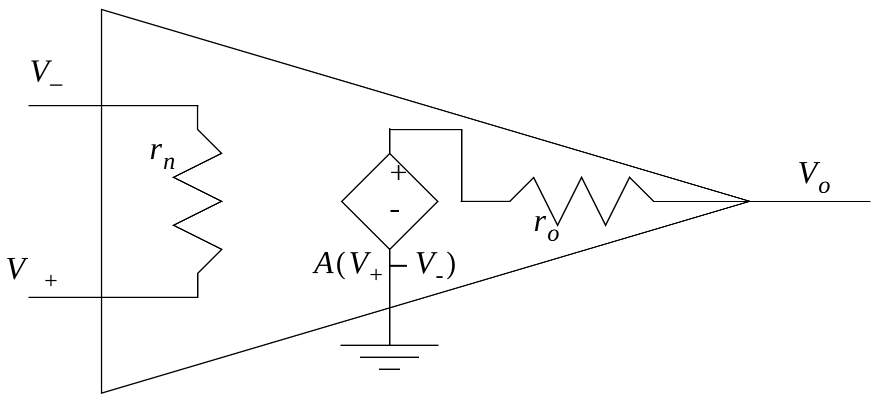
$$V_o\left(\frac{R_1}{R_1 + R_2}\right) = V_i\left(\frac{R_2}{R_1 + R_2}\right) - V_{ref}\left(\frac{R_4}{R_4 + R_5}\right)$$

$$V_o = V_i\left(\frac{R_2}{R_1}\right) - V_{ref}\left(\frac{R_4(R_1 + R_2)}{R_1(R_4 + R_5)}\right) \qquad (3)$$

Figure 7.18 Op-amp example (continued)

An op-amp (operational amplifier) has an extremely high gain, typically 100,000 times. The gain is multiplied by the difference between the inverting and non-inverting terminals to form an output. A typical op-amp will work for signals from DC up to about

100KHz. When the op-amp is being used for high frequencies or large gains, the model of the op-amp in Figure 7.19 should be used. This model includes a large resistance between the inverting and non-inverting inputs. The voltage difference drives a dependent voltage source with a large gain. The output resistance will limit the maximum current that the device can produce, normally less than 100mA.



typically,
$r_n > 1M\Omega$
$A > 10^5$
$r_o < 100\Omega$

Figure 7.19 A non-ideal op-amp model

7.3 IMPEDANCE

Circuit components can be represented in impedance form as shown in Figure 7.20. When represented this way the circuit solutions can focus on impedances, ’Z’, instead of resistances, ’R’. Notice that the primary difference is that the differential operator has been replaced. In this form we can use impedances as if they are resistances.

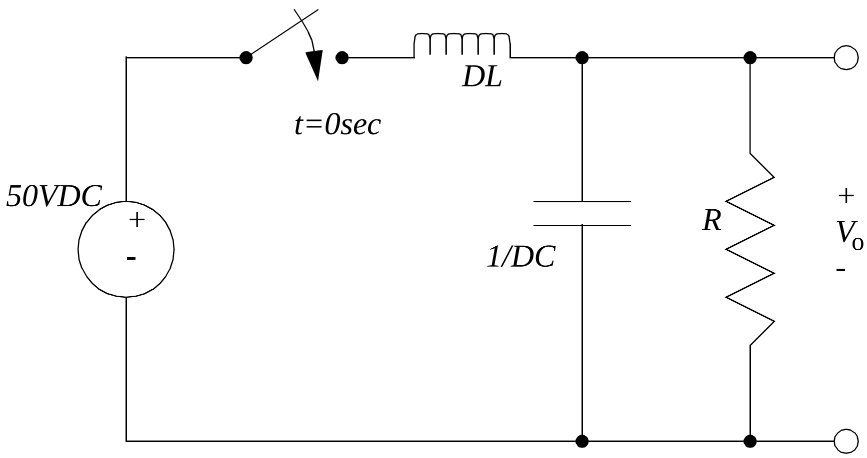
Device	*Time domain*	*Impedance*
Resistor	$V(t) = RI(t)$	$Z = R$
Capacitor	$V(t) = \frac{1}{C}\int I(t)dt$	$Z = \frac{1}{DC}$
Inductor	$V(t) = L\frac{d}{dt}I(t)$	$Z = LD$

Note: Impedance is like resistance, except that it includes time variant features also.

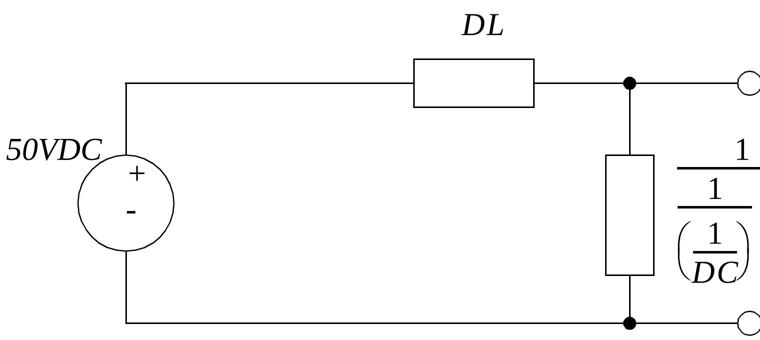
$$V = ZI$$

Figure 7.20 Impedances for electrical components

When representing component values with impedances the circuit solution is done as if all circuit components are resistors. An example of this is shown in Figure 7.21. Notice that the two impedances at the right (resistor and capacitor) are equivalent to two resistors in parallel, and the overall circuit is a voltage divider. The impedances are written beside the circuit elements.



Find the equivalent for the capacitor and resistor in parallel.



$$\frac{1}{\frac{1}{\left(\frac{1}{DC}\right)} + \frac{1}{R}} = \frac{1}{DC + \frac{1}{R}} = \frac{R}{RCD + 1}$$

Treat the circuit as a voltage divider,

$$V_o = 50V\frac{\left(\frac{R}{1 + DCR}\right)}{DL + \left(\frac{R}{1 + DCR}\right)} = 50V\left(\frac{R}{D^2RLC + DL + R}\right)$$

Figure 7.21 A impedance example for a circuit

7.4 EXAMPLE SYSTEMS

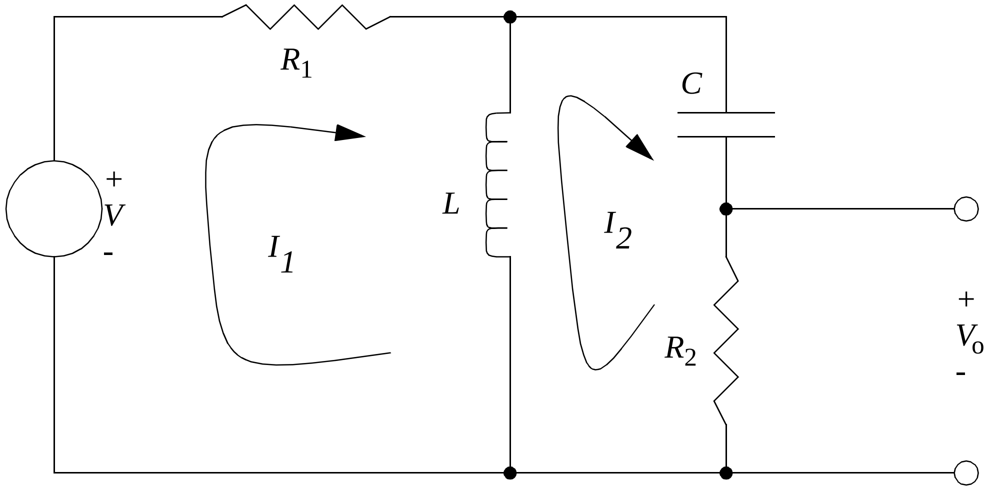
The list of instructions below can be useful when approaching a circuits problem. The most important concept to remember is that a minute of thinking about the solution approach will save ten minutes of backtracking and fixing mistakes.

1. Look at the circuit to determine if it is a standard circuit type such as a voltage divider, current divider or an op-amp inverting amplifier. If so, use the standard solution to solve the problem.
2. Otherwise, consider the nodes and loops in the circuit. If the circuit contains fewer loops, select the current loop method. If the circuit contains fewer nodes, select the node voltage method. Before continuing, verify that the select method can be used for the circuit.
3. For the node voltage method define node voltages and current directions. For the current loop method define current loops and indicate voltage rises or drops by adding ’+’ or ’-’ signs.
4. Write the equations for the loops or nodes.
5. Identify the desired value and eliminate unwanted values using algebra techniques.
6. Use numerical values to find a final answer.

Note: The units for various electrical quantities are listed to the right. They may be used to check equations by doing a unit balance.

coefficient	units
C	$\frac{As}{V}$
L	$\frac{Vs}{A}$
R	$\frac{V}{A}$

The circuit in Figure 7.22 could be solved with two loops, or two nodes. An arbitrary decision is made to use the current loop method. The voltages around each loop are summed to provide equations for each loop.



Note: when summing voltages in a loop remember to deal with sources that increase the voltage by flipping the sign.

First, sum the voltages around the loops and then eliminate I1.

$$\sum V_{L1} = -V + R_1 I_1 + L(DI_1 - DI_2) = 0$$

$$(R_1 + LD)I_1 = V + (LD)I_2 \qquad (1)$$

$$I_1 = \frac{V}{R_1 + LD} + \left(\frac{LD}{R_1 + LD}\right)I_2 \qquad (2)$$

$$\sum V_{L2} = L(DI_2 - DI_1) + \frac{I_2}{CD} + R_2 I_2 = 0$$

$$L(DI_2 - DI_1) + \frac{I_2}{CD} + R_2 I_2 = 0 \qquad (3)$$

$$(DL)I_1 = \left(LD + \frac{1}{CD} + R_2\right)I_2$$

$$I_1 = \left(1 + \frac{1}{CLD^2} + \frac{R_2}{LD}\right)I_2 \qquad (4)$$

Figure 7.22 Example problem

The equations in Figure 7.22 are manipulated further in Figure 7.23 to develop an input-output equation for the second current loop. This current can be used to find the current through the output resistor R_2. The output voltage can then be found by multiplying the R_2 and I_2.

First, sum the voltages around the loops and then eliminate I_1.

$$I_1 = \frac{V}{R_1 + LD} + \left(\frac{LD}{R_1 + LD}\right) I_2 = \left(1 + \frac{1}{CLD^2} + \frac{R_2}{LD}\right) I_2$$

$$\frac{V}{R_1 + LD} = \left(1 + \frac{1}{CLD^2} + \frac{R_2}{LD} - \frac{LD}{R_1 + LD}\right) I_2$$

$$\frac{V}{R_1 + LD} = \left(\frac{(CLD^2 + 1 + CDR_2)(R_1 + LD) - CL^2D^3}{CLD^2}\right) I_2$$

$$I_2 = \left(\frac{CLD^2}{(R_1 + LD)((CLD^2 + 1 + CDR_2)(R_1 + LD) - CL^2D^3)}\right) V$$

$$I_2 = \left(\frac{CLD^2}{CL(R_1 + R_2)D^3 + L(CR_1^2 + 2CR_1R_2 + L)D^2 + R_1(CR_1R_2 + 2L)D + (R_1^2)}\right) V$$

Convert it to a differential equation.

$$CL(R_1 + R_2)I_2''' + L(CR_1^2 + 2CR_1R_2 + L)I_2'' + R_1(CR_1R_2 + 2L)I_2' + (R_1^2)I_2 = CLV''$$

Figure 7.23 Example problem (continued)

The equations can also be manipulated into state equations, as shown in Figure 7.24. In this case a dummy variable is required to replace the two first derivatives in the first equation. The dummy variable is used in place of I_1, which now becomes an output variable. In the remaining state equations I_1 is replaced by q_1. In the final matrix form the state equations are in one matrix, and the output variable must be calculated separately.

State equations can also be developed using equations (1) and (3).

(1) becomes $R_1 I_1 + L\dot{I}_1 = V + L\dot{I}_2$

$$L\dot{I}_1 - L\dot{I}_2 = V - R_1 I_1$$

$$\dot{I}_1 - \dot{I}_2 = \frac{V}{L} - \frac{R_1}{L} I_1$$

$$q_1 = I_1 - I_2 \qquad (10)$$

$$\dot{q}_1 = \frac{V}{L} - \frac{R_1}{L} I_1$$

$$I_1 = q_1 + I_2 \qquad (11)$$

$$\dot{q}_1 = \frac{V}{L} - \frac{R_1}{L}(q_1 + I_2)$$

$$\dot{q}_1 = q_1\left(-\frac{R_1}{L}\right) + I_2\left(-\frac{R_1}{L}\right) + \frac{V}{L} \qquad (12)$$

(3) becomes $L\ddot{I}_2 - L\ddot{I}_1 + \frac{I_2}{C} + R_2\dot{I}_2 = 0$

$$V - R_1 I_1 + \frac{I_2}{C} + R_2\dot{I}_2 = 0$$

$$V - R_1(q_1 + I_2) + \frac{I_2}{C} = -R_2\dot{I}_2$$

$$\dot{I}_2 = I_2\left(-R_1 + \frac{1}{C}\right) + q_1(-R_1) + V \qquad (13)$$

These can be put in matrix form,

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{R_1}{L} \\ -R_1 & -R_1 + \frac{1}{C} \end{bmatrix}\begin{bmatrix} q_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} \frac{V}{L} \\ V \end{bmatrix} \qquad \begin{bmatrix} I_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ I_2 \end{bmatrix}$$

Figure 7.24 Example problem (continued)

Evaluate the circuit in Figure 7.22 using the node voltage method.

Figure 7.25 Drill problem: Use the node voltage method

Find the equation relating the output and input voltages,

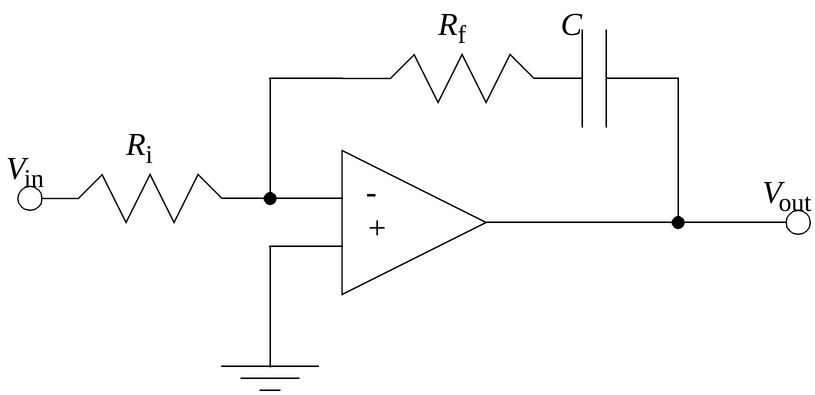
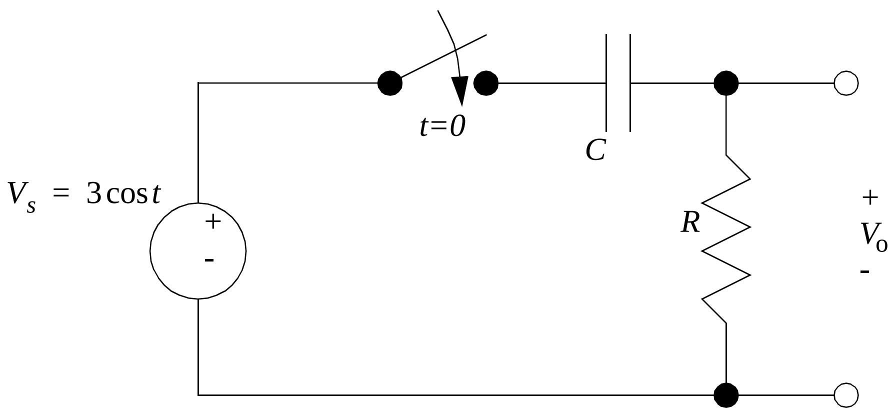


Figure 7.26 Drill problem: Find the state equation

The circuit in Figure 7.27 can be evaluated as a voltage divider when the capacitor is represented as an impedance. In this case the result is a first-order differential equation.



As normal we relate the source voltage to the output voltage. The we find the values for the various terms in the frequency domain.

$$V_o = V_s\left(\frac{Z_R}{Z_R + Z_C}\right) \quad \text{where,} \quad Z_R = R \quad Z_C = \frac{1}{DC}$$

Next, we may combine the equations, and convert it to a differential equation.

$$V_o = V_s\left(\frac{R}{R + \frac{1}{DC}}\right)$$

$$V_o = V_s\left(\frac{CRD}{CRD + 1}\right)$$

$$V_o(CRD + 1) = V_s(CRD)$$

$$\dot{V}_o(CR) + V_o = \dot{V}_s(CR)$$

$$\dot{V}_o + V_o\left(\frac{1}{CR}\right) = \dot{V}_s$$

Figure 7.27 Circuit solution using impedances

The first-order differential equation in Figure 7.27 is continued in Figure 7.28 where the equation is integrated. The solution is left in variable form, except for the supply voltage.

First, write the homogeneous solution using the known relationship.

$$\dot{V}_o + V_o\left(\frac{1}{CR}\right) = 0 \qquad \text{yields} \qquad V_h = C_1 e^{-\frac{t}{CR}}$$

Next, the particular solution can be determined, starting with a guess.

$$\dot{V}_o + V_o\left(\frac{1}{CR}\right) = \left(\frac{d}{dt}\right)(3\cos t) = -3\sin t$$

$$V_p = A\sin t + B\cos t$$

$$V_p' = A\cos t - B\sin t$$

$$(A\cos t - B\sin t) + (A\sin t + B\cos t)\left(\frac{1}{CR}\right) = -3\sin t$$

$$A + B\left(\frac{1}{CR}\right) = 0$$

$$A = B\left(\frac{-1}{CR}\right)$$

$$-B + A\left(\frac{1}{CR}\right) = -3$$

$$-B + B\left(\frac{-1}{CR}\right)\left(\frac{1}{CR}\right) = -3$$

$$B\left(\frac{1}{C^2R^2} + 1\right) = 3$$

$$B = \frac{3C^2R^2}{1 + C^2R^2} \qquad A = \left(\frac{3C^2R^2}{1 + C^2R^2}\right)\left(\frac{-1}{CR}\right) = \frac{-3CR}{1 + C^2R^2}$$

$$V_p = \sqrt{A^2 + B^2}\sin\left(t + \operatorname{atan}\left(\frac{B}{A}\right)\right)$$

The homogeneous and particular solutions can now be combined. The system will be assumed to be at rest initially.

$$V_0 = V_h + V_p = C_1 e^{-\frac{t}{CR}} + \sqrt{A^2 + B^2}\sin\left(t + \operatorname{atan}\left(\frac{B}{A}\right)\right)$$

$$0 = C_1 e^0 + \sqrt{A^2 + B^2}\sin\left(0 + \operatorname{atan}\left(\frac{B}{A}\right)\right)$$

$$C_1 = -\sqrt{A^2 + B^2}\sin\left(0 + \operatorname{atan}\left(\frac{B}{A}\right)\right)$$

Figure 7.28 Circuit solution using impedances (continued)

7.5 ELECTROMECHANICAL SYSTEMS - MOTORS

7.5.1 Permanent Magnet DC Motors

DC motors apply a torque between the rotor and stator that is related to the applied voltage/current. When a voltage is applied the torque will cause the rotor to accelerate. For any voltage and load on the motor there will tend to be a final angular velocity due to friction and drag in the motor. And, for a given voltage the ratio between steady-state torque and speed will be a straight line, as shown in Figure 7.29.

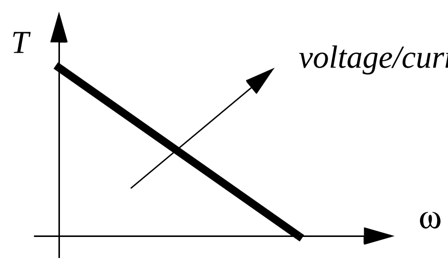
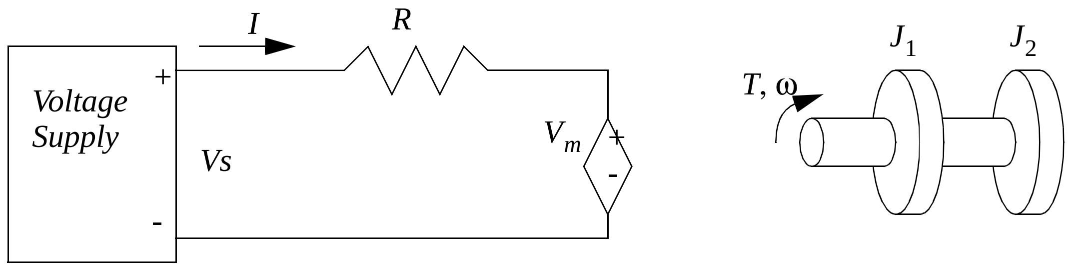


Figure 7.29 Torque speed curve for a permanent magnet DC motor

The basic equivalent circuit model is shown in Figure 7.30, includes the rotational inertia of the rotor and any attached loads. On the left hand side is the resistance of the motor and the ’back emf’ dependent voltage source. On the right hand side the inertia components are shown. The rotational inertia J1 is the motor rotor, and the second inertia is an attached disk.



Because a motor is basically wires in a magnetic field, the electron flow (current) in the wire will push against the magnetic field. And, the torque (force) generated will be proportional to the current.

$$T_m = KI \qquad \therefore I = \frac{T_m}{K}$$

Next, consider the power in the motor,

$$P = V_m I = T\omega = KI\omega \qquad \therefore V_m = K\omega$$

Consider the dynamics of the rotating masses by summing moments.

$$\sum M = T_m - T_{load} = J\left(\frac{d}{dt}\right)\omega \qquad \therefore T_m = J\left(\frac{d}{dt}\right)\omega + T_{load}$$

Figure 7.30 The torque and inertia in a basic motor model

These basic equations can be manipulated into the first-order differential equation in Figure 7.31.

The current-voltage relationship for the left hand side of the equation can be written and manipulated to relate voltage and angular velocity.

$$I = \frac{V_s - V_m}{R}$$

$$\therefore \frac{T_m}{K} = \frac{V_s - K\omega}{R}$$

$$\therefore \frac{J\left(\frac{d}{dt}\right)\omega + T_{load}}{K} = \frac{V_s - K\omega}{R}$$

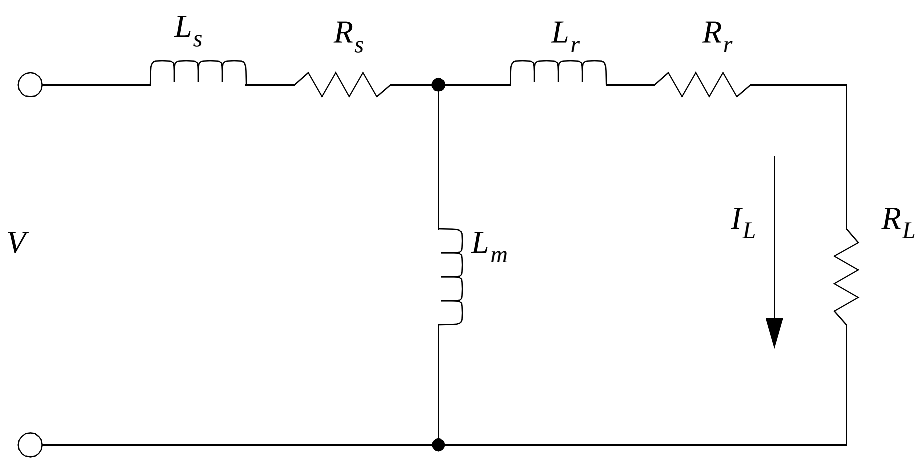
$$\boxed{\therefore \left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{JR}\right) = V_s\left(\frac{K}{JR}\right) - \frac{T_{load}}{J}}$$

Figure 7.31 The first-order model of a motor

7.5.2 Induction Motors

AC induction motors are extremely common because of the low cost of construction, and compatibility with the power distribution system. The motors are constructed with windings in the stator (outside of the motor). The rotor normally has windings, or a squirrel cage. The motor does not have bushes to the rotor. The motor speed is close to, but always less than the rotating AC fields. The rotating fields generate currents, and hence opposing magnetic fields in the stator. The maximum motor speed is a function of the frequency of the AC power, and the number of pole of the machine. For example, an induction motor with three poles being used with a 60Hz AC supply would have a maximum speed of 2*(60Hz/3) = 40Hz = 2400RPM.

The equivalent circuit for an AC motor is given in Figure 7.32. The slip of the motor determines the load current, I_L. It is a function of the fraction, f, of full speed.



$$R_L = \frac{1-f}{f} R_r$$

Figure 7.32 Basic model of an induction motor

The torque relationship for AC motors is given in Figure 7.33. These can be combined with the equivalent circuit model to determine the response of the motor to a load.

First the torques on the motor are summed,

$$\sum M = T_{rotor} - T_{load} = J\frac{d}{dt}\omega$$

CONTINUE TO STATE VARIABLE MODEL....

7.5.3 Brushless Servo Motors

Brushless servo motors are becoming very popular because of their low maintenance requirements. The motors eliminate the need for brushes by using permanent magnets on the rotor, with windings on the stator, as shown in Figure 7.33. The windings on the stator are switched at a given frequency to produce a desired rotational speed, or held static to provide a holding torque.

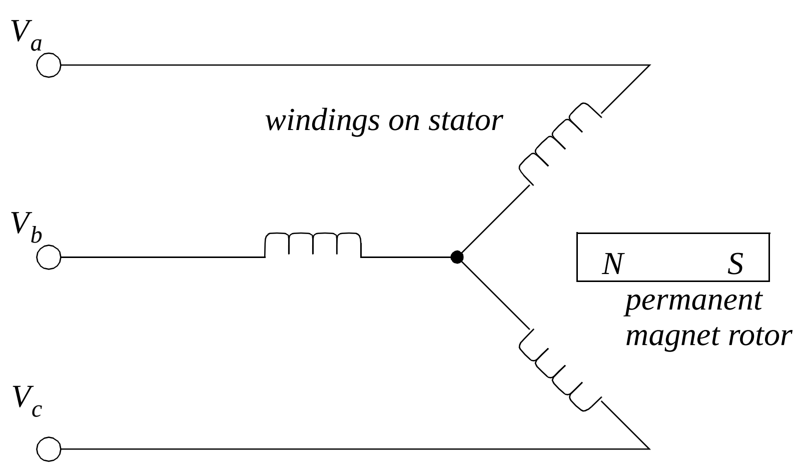


Figure 7.33 The construction of a brushless servo motor

The basic relationships for brushless DC motors are given in Figure 7.34.

$$V_t = \left(R_m + \frac{d}{dt}L\right)I_m + E$$

$$E = K_e\omega$$

$$T = K_t I_m$$

where,

V_t = *terminal voltage across motor windings*
R_m = *resistance of a motor winding*
L = *phase to phase inductance*
I_m = *current in winding*
E = *back e.m.f. of motor*
K_e = *motor speed constant*
ω = *motor speed*
K_t = *motor torque constant*
T = *motor torque*

Figure 7.34 Basic relationships for a brushless motor

$$V_t = \left(R_m + \frac{d}{dt}L\right)\frac{T}{K_t} + K_e\omega$$

$$\sum M = T - T_{load} = J\frac{d}{dt}\omega$$

$$T = J\frac{d}{dt}\omega + T_{load}$$

where,

J = *combined moments of inertia for the rotor and external loads*

T_{load} = *the applied torque in the system*

$$V_t = \left(R_m + \frac{d}{dt}L\right)\frac{J\frac{d}{dt}\omega + T_{load}}{K_t} + K_e\omega$$

$$V_t = \frac{JR_m}{K_t}\frac{d}{dt}\omega + \frac{LJ}{K_t}\left(\frac{d}{dt}\right)^2\omega + \frac{R_m}{K_t}T_{load} + \frac{L}{K_t}\frac{d}{dt}T_{load} + K_e\omega$$

$$(LJ)\left(\frac{d}{dt}\right)^2\omega + (JR_m)\frac{d}{dt}\omega + K_eK_t\omega = K_tV_t - L\frac{d}{dt}T_{load} - R_mT_{load}$$

$$\left(\frac{d}{dt}\right)^2\omega + \frac{R_m}{L}\frac{d}{dt}\omega + \frac{K_eK_t}{LJ}\omega = \frac{K_tV_t}{LJ} - \frac{1}{J}\frac{d}{dt}T_{load} - \frac{R_mT_{load}}{LJ}$$

Figure 7.35 An advanced model of a brushless servo motor

To rotate the motor at a constant velocity the waveform in Figure 7.36 would be applied to each phase. Although each phase would be 120 degrees apart for a three pole motor. A more sophisticated motor controller design would smooth the waves more to approach a sinusoidal shape.

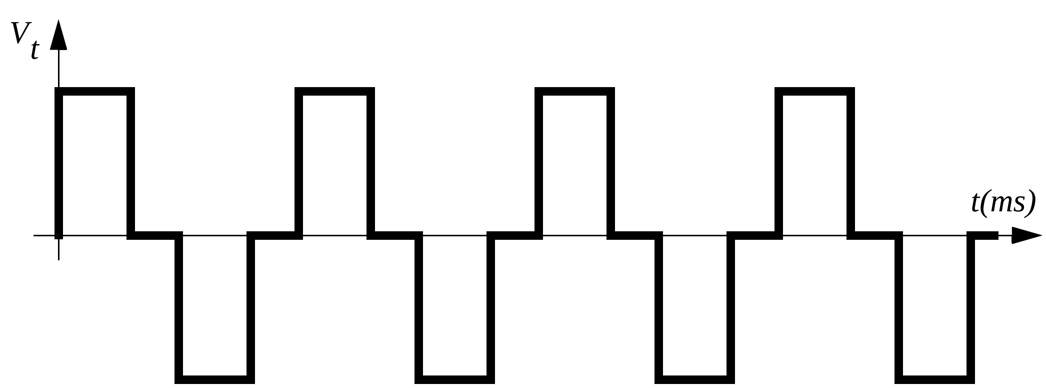


Figure 7.36 Typical supply voltages

7.6 FILTERS

Filters are useful when processing data signals. Low pass often used to eliminate noise, high pass filters eliminate static signals and leave dynamic signals. Band pass filters reject all frequencies outside a desired frequency band. A low pass filter is shown in Figure 7.37. At high frequencies the capacitor, C, has a very low impedance, and grounds the input signal. At low frequencies the capacitor impedance is high, increasing the gain of the op-amp circuit. This is easier to conceptualize if the R_1-C pair are viewed as a voltage divider.

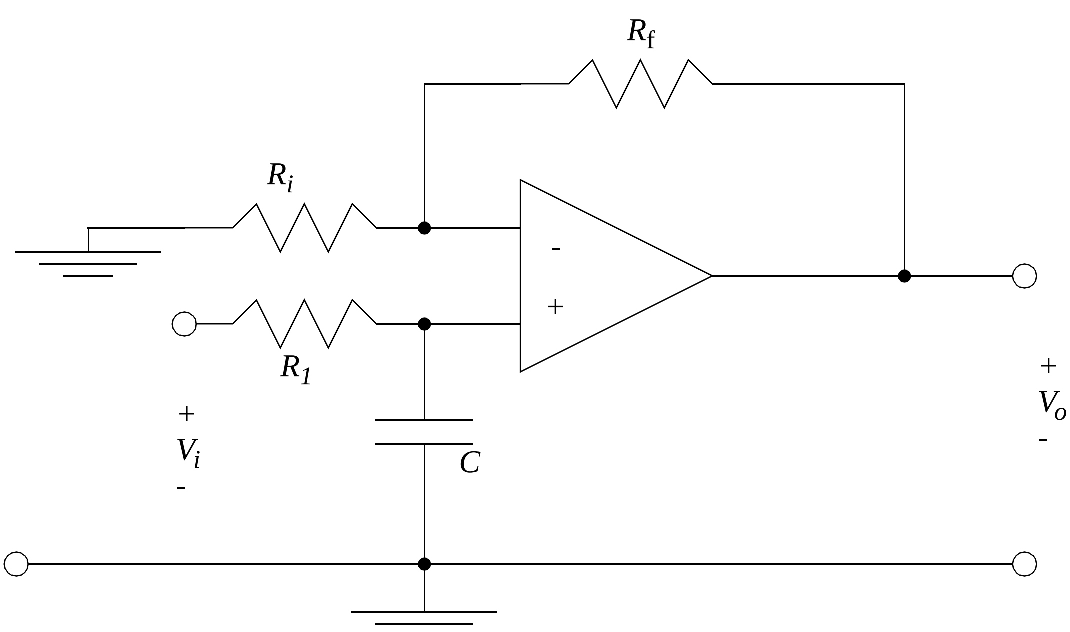


Figure 7.37 Low-Pass Filter

A high pass filter is shown in Figure 7.38. In this case the voltage divider in the previous circuit is reversed. In this circuit the gain will increase for signals with higher frequencies.

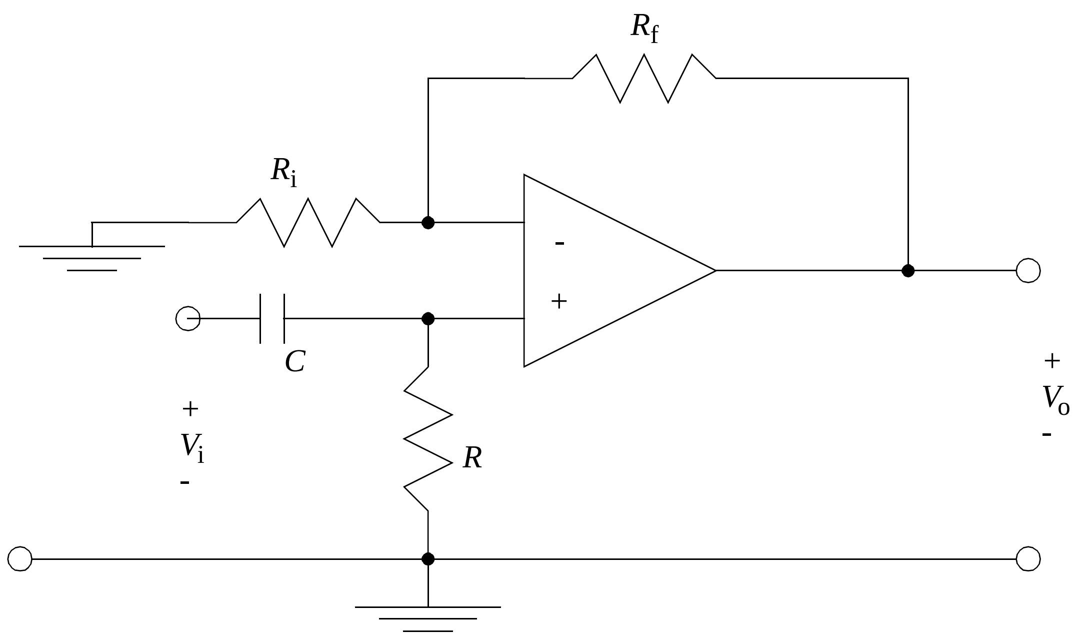


Figure 7.38 High-Pass Filter

7.7 OTHER TOPICS

The relationships in Figure 7.39 can be used to calculate the power and energy in a system. Notice that the power calculations focus on resistance, as resistances will dissipate power in the form of heat. Other devices, such as inductors and capacitors, store energy, but don't dissipate it.

$$P = IV = I^2 R = \frac{V^2}{R} \qquad E = Pt$$

Figure 7.39 Electrical power and energy

7.8 SUMMARY

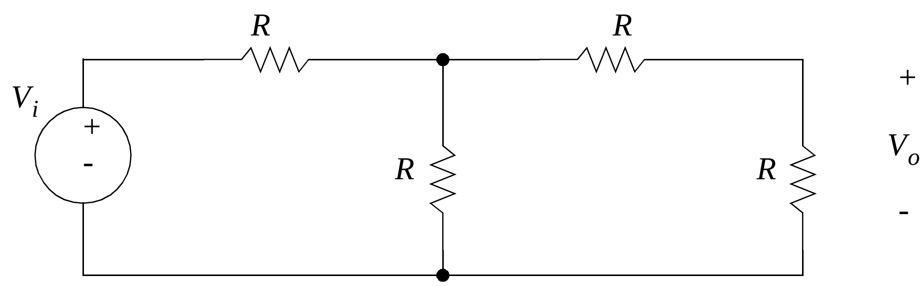
- Basic circuit components are resistors, capacitors, inductors op-amps.
- node and loop methods can be used to analyze circuits.

• Capacitor and inductor impedances can be used as resistors in calculations.

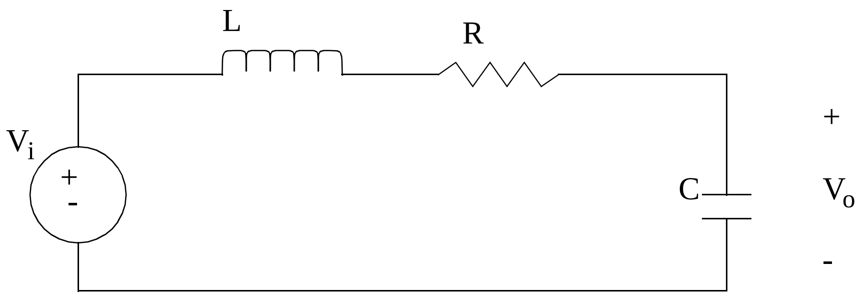
7.9 PRACTICE PROBLEMS

1. Derive the equations for combined values for resistors, capacitors and inductors in series and parallel.

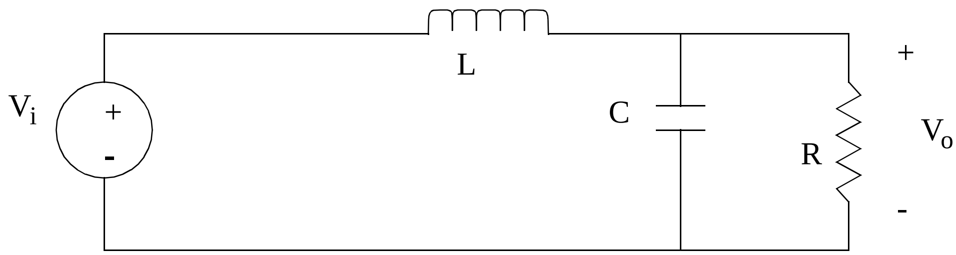
2. Find the output voltage as a function of input voltage.



3. Write the differential equation for the following circuit.

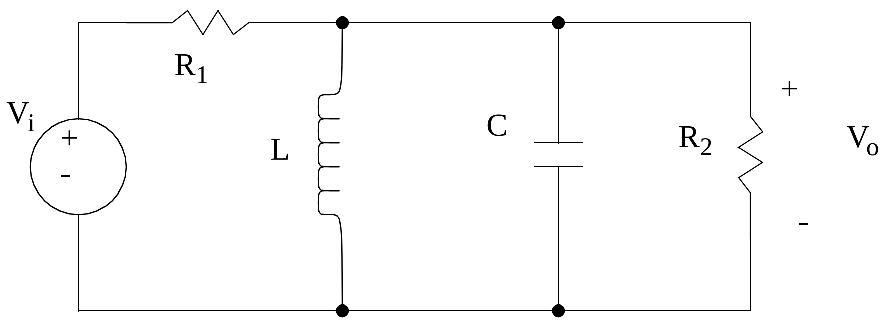


4. Consider the following circuit.

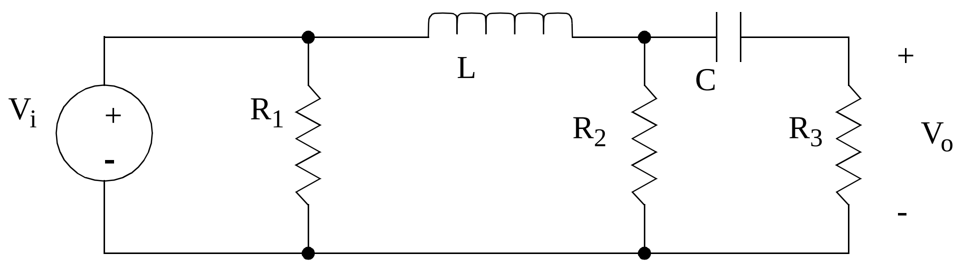


a) Develop a differential equation for the circuit.
b) Put the equation in state variable matrix form.

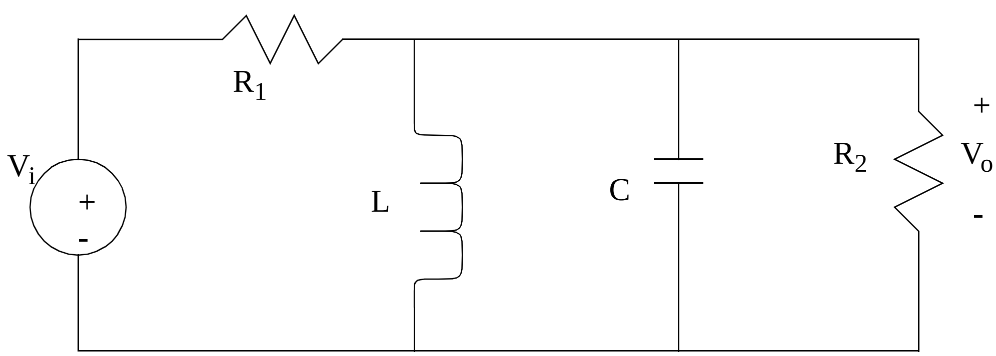
5. Develop differential equations and the input-output equation for the electrical system below.



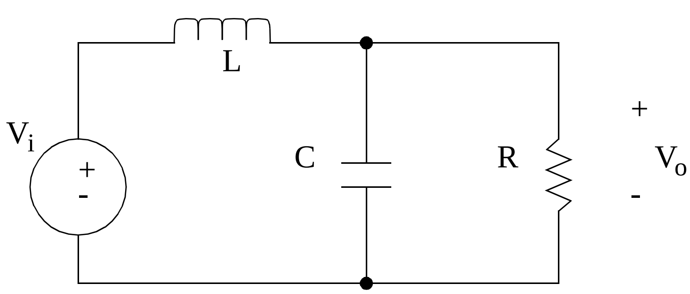
6. Consider the following circuit. Develop a differential equation for the circuit.



7. Find the input-output equation for the circuit below, and then find the natural frequency and damping coefficient.



8. a) Find the differential equation for the circuit below where the input is Vi, and the output is Vo.



b) Convert the equation to an input-output equation.

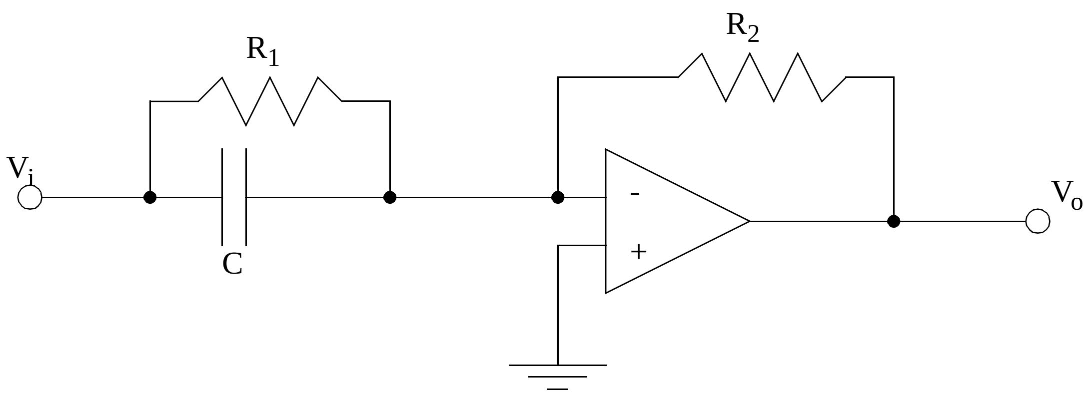
c) Solve the differential equation found in part b) using the numerical values given below. Assume at time t=0, the circuit has the voltage Vo and the first derivative shown below.

$$L = 10mH \qquad C = 1\mu F \qquad R = 1K\Omega \qquad V_i = 10V$$

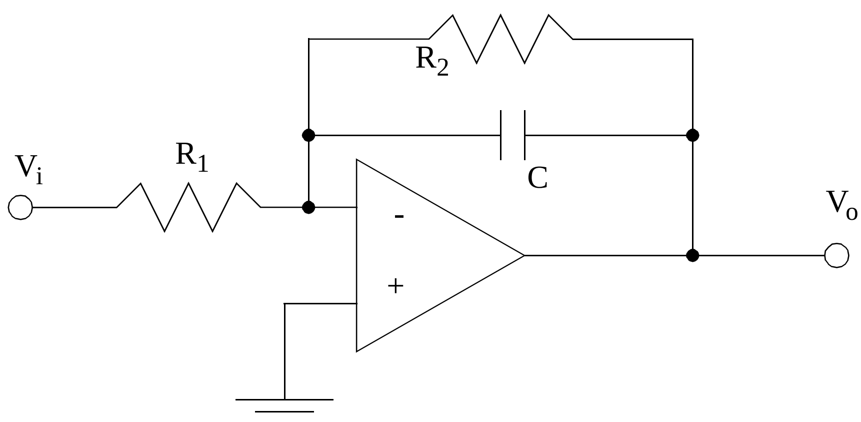
at t=0s $\quad V_o = 2V \qquad V_o' = 3\frac{V}{s}$

9.

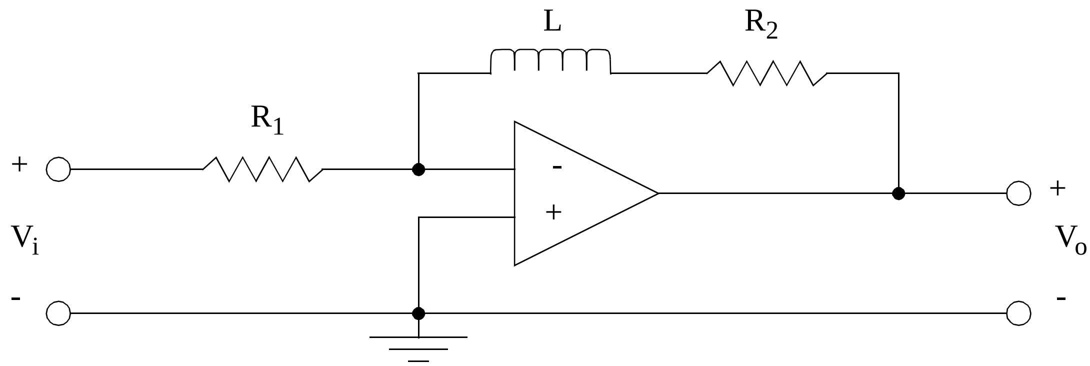
a) Write the differential equations for the system pictured below.
b) Put the equations in input-output form.



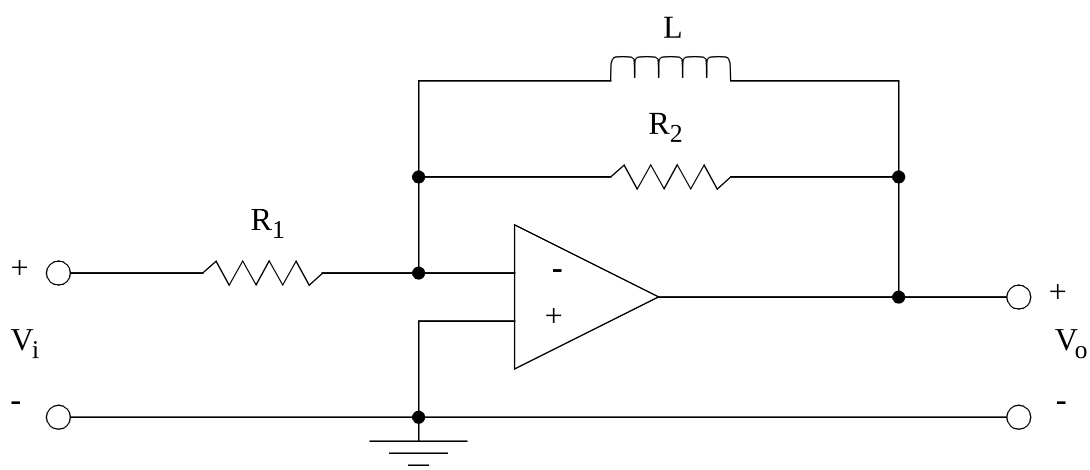
10. Given the circuit below, find the ratio of the output over the input (this is also known as a transfer function). Simplify the results.



11. Examine the following circuit and then derive the differential equation.

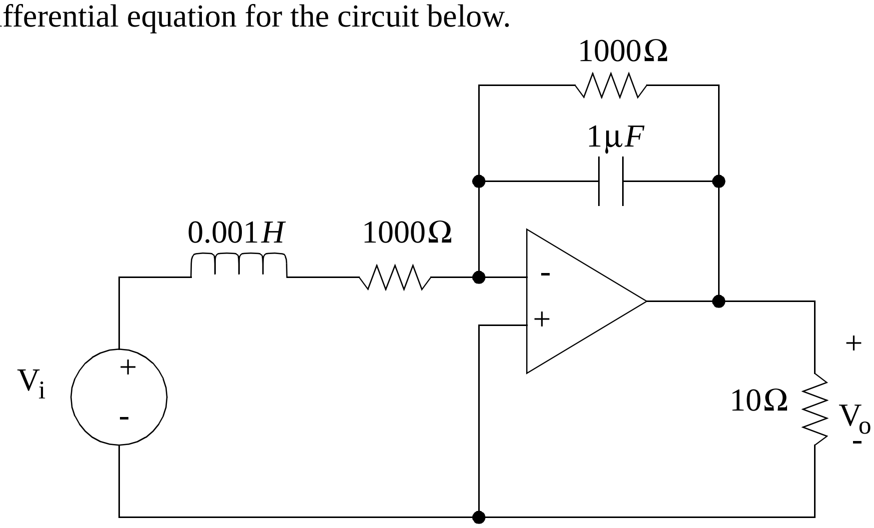


12. Examine the following circuit and then derive the differential equation.



13.

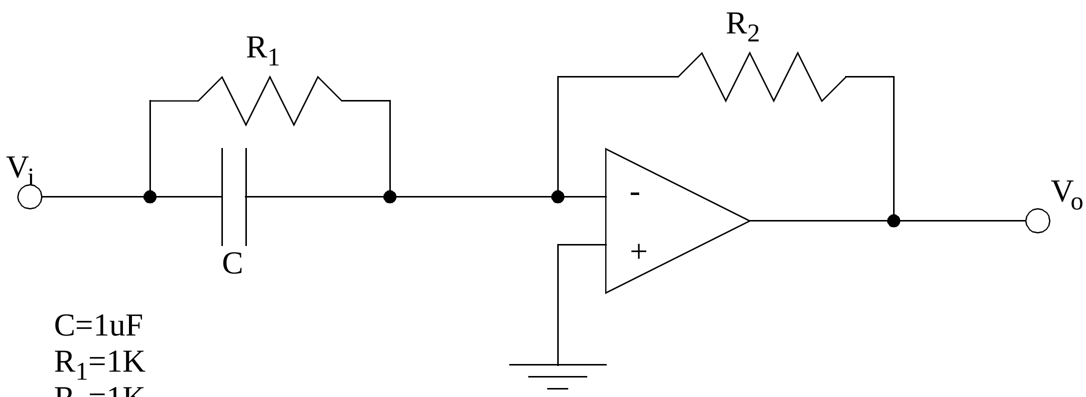
a) Find the differential equation for the circuit below.



b) Put the differential equation in state variable form and a numerical method to produce a detailed sketch of the output voltage Vo. Assume the system starts at rest, and the input is Vi=5V.

14.

a) Write the differential equations for the system pictured below.
b) Put the equations in state variable form.
c) Use numerical methods to find the ratio between input and output voltages for a range of frequencies. The general method is put in a voltage such as Vi=1sin(___t), and see what the magnitude of the output is. Divide the magnitude of the output sine wave by the input magnitude. Note: This should act as a high pass or low pass filter.
d) Plot a graph of gain against the frequency of the input.



C=1uF
R_1=1K
R_2=1K

7.10 PRACTICE PROBLEM SOLUTIONS

1.

$$R_{parallel} = \frac{R_1 R_2}{R_1 + R_2} \qquad R_{series} = R_1 + R_2$$

$$C_{series} = \frac{C_1 C_2}{C_1 + C_2} \qquad C_{parallel} = C_1 + C_2$$

$$L_{parallel} = \frac{L_1 L_2}{L_1 + L_2} \qquad L_{series} = L_1 + L_2$$

2.

$$\frac{V_o}{V_i} = \frac{1}{5}$$

3.

$$\ddot{V}_o + \dot{V}_o\left(\frac{R}{L}\right) + V_o\left(\frac{1}{LC}\right) = V_i\left(\frac{1}{LC}\right)$$

4.

a) $$\ddot{V}_o + \dot{V}_o\left(\frac{1}{RC}\right) + V_o\left(\frac{1}{LC}\right) = V_i\left(\frac{1}{LC}\right)$$

b) $$\dot{V}_o = X_o$$

$$\dot{X}_o = X_o\left(\frac{-1}{RC}\right) + V_o\left(\frac{-1}{LC}\right) + V_i\left(\frac{1}{LC}\right)$$

5.

$$\ddot{V}_o + \dot{V}_o\left(\frac{1}{CR_1} + \frac{1}{CR_2}\right) + V_o\left(\frac{1}{LC}\right) = \dot{V}_i\left(\frac{1}{CR_1}\right)$$

6.

$$\ddot{V}_o + \dot{V}_o\left(\frac{R_2R_3C + L}{LC(R_2 + R_3)}\right) + V_o\left(\frac{R_2}{LC(R_2 + R_3)}\right) = \dot{V}_i\left(\frac{R_2R_3}{L(R_2 + R_3)}\right)$$

7.

a) $$\ddot{V}_o + \dot{V}_o\left(\frac{1}{CR_1} + \frac{1}{CR_2}\right) + V_o\left(\frac{1}{LC}\right) = \dot{V}_i\left(\frac{1}{CR_1}\right)$$

b) $$\omega_n = \sqrt{\frac{1}{LC}} \qquad \zeta = \frac{\sqrt{L}(R_1 + R_2)}{2\sqrt{C}R_1R_2}$$

8.

(a.

Sum currents at node Vo

$$\sum I_{V_o} = \frac{(V_o - V_i)}{DL} + (V_o)DC + \frac{(V_o)}{R} = 0$$

$$V_o - V_i + V_o D^2 LC + \frac{V_o DL}{R} = 0$$

$$V_o'' + V_o'\left(\frac{1}{CR}\right) + V_o\left(\frac{1}{LC}\right) = V_i\left(\frac{1}{LC}\right)$$

(b.

$$\ddot{V}_o + \dot{V}_o\left(\frac{1}{CR}\right) + V_o\left(\frac{1}{LC}\right) = V_i\left(\frac{1}{LC}\right)$$

(c.

$$\ddot{V}_o(LC) + \dot{V}_o\left(\frac{L}{R}\right) + V_o = V_i \qquad \ddot{V}_o(10^{-2}10^{-6}) + \dot{V}_o\left(\frac{10^{-2}}{10^{3}}\right) + V_o = 10$$

$$\ddot{V}_o(10^{-8}) + \dot{V}_o(10^{-5}) + V_o = 10$$

$$\ddot{V}_o + \dot{V}_o(10^{3}) + V_o(10^{8}) = (10^{9})$$

homogeneous;

$$A^2e^{At} + Ae^{At}(10^3) + e^{At}(10^8) = 0$$

$$A^2 + A(10^3) + (10^8) = 0 \qquad A = \frac{-10^3 \pm \sqrt{(10^3)^2 - 4(10^8)}}{2} = -500 \pm 9987j$$

$$V_h = C_1e^{-500t}\cos(9987t + C_2)$$

particular; guess

$$V_p = A \qquad (0) + (0)(10^3) + (A)(10^8) = (10^9) \qquad A = 10$$

$$V_p = 10$$

for initial conditions,

$$V_o = C_1e^{-500t}\cos(9987t + C_2) + 10$$

for t=0, Vo=2V

$$2 = C_1e^{-500(0)}\cos(9987(0) + C_2) + 10$$

$$-8 = C_1\cos(C_2) \qquad (1)$$

$$\dot{V}_o = -500C_1e^{-500t}\cos(9987t + C_2) - 9987C_1e^{-500t}\sin(9987t + C_2)$$

for t=0, d/dt Vo=3V

$$3 = -500C_1\cos(C_2) - 9987C_1\sin(C_2)$$

$$3 = -500C_1\cos(C_2) - 9987C_1\sin(C_2)$$

$$3 = -4000 - 9987\left(\frac{-8}{\cos(C_2)}\right)\sin(C_2)$$

$$\frac{-3997}{8(9987)} = \frac{\sin(C_2)}{\cos(C_2)} = \tan(C_2) \qquad C_2 = -0.050$$

$$C_1 = \frac{-8}{\cos(C_2)} = -8.01$$

$$V_o = -8.01e^{-500t}\cos(9987t - 0.050) + 10$$

9.

a) $$V_i + \dot{V}_i(R_1 C) + V_o\left(\frac{R_1}{R_2}\right) = 0$$

b) $$V_o = \dot{V}_i(-CR_2) + V_i\left(\frac{-R_2}{R_1}\right)$$

10.

$$\frac{V_o}{V_i} = \frac{-R_2}{R_1 + DR_1R_2C}$$

11.

$$V_o = \dot{V}_i\left(\frac{-L}{R_1}\right) + V_i\left(\frac{-R_2}{R_1}\right)$$

12.

$$\dot{V}_o + V_o\left(\frac{R_2}{L}\right) = \dot{V}_i\left(\frac{-R_2}{R_1}\right)$$

13.

(a.

Create a node between the inductor and resistor Va, and use the node voltage method

$$\sum I_{V_A} = \frac{(V_A - V_i)}{0.001D} + \frac{(V_A - V_-)}{1000} = 0$$

$$V_- = V_+ = 0V$$

$$1000000(V_A - V_i) + V_A D = 0$$

XXXXADD UNITSXXXXX

$$V_A = V_i\left(\frac{1000000}{1000000 + D}\right)$$

$$\sum I_{V_-} = \frac{(V_- - V_A)}{1000} + \frac{(V_- - V_o)}{1000} + (V_- - V_o)(0.000001D) = 0$$

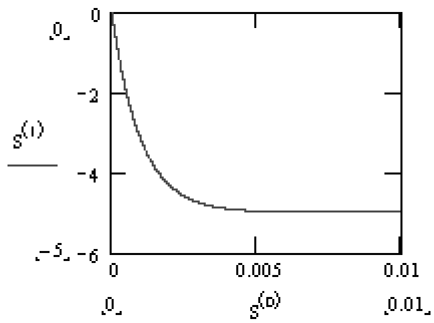
$$\frac{(-1)}{1000}V_i\left(\frac{1000000}{1000000 + D}\right) + \frac{(-V_o)}{1000} + (-V_o)(0.000001D) = 0$$

$$V_o(-1 - 0.001D) = V_i\left(\frac{1000000}{1000000 + D}\right)$$

$$V_o(-1000000 - D - 1000D - 0.001D^2) = 1000000V_i$$

$$V_o''(-10^{-9}) + V_o'(-1.001(10^{-3})) + V_o(-1) = V_i$$

(b. $\frac{d}{dt}V_o = \dot{V}_o$ $\frac{d}{dt}\dot{V}_o = -1000000000V_i - 1001000\dot{V}_o - 1000000000V_o$



14.

a) $V_o = \dot{V}_i(-CR_2) + V_i\left(\frac{-R_2}{R_1}\right)$

b) Not a state equation

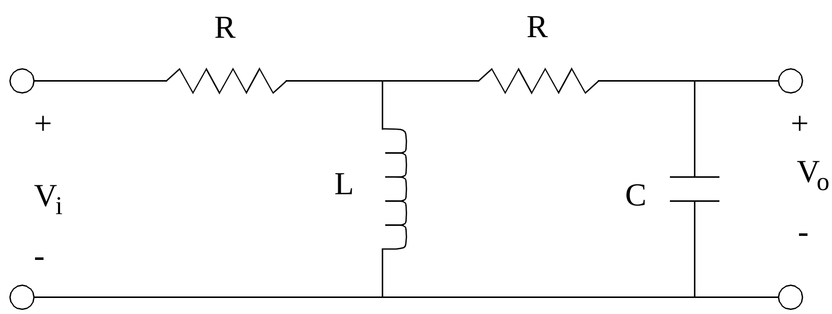
c) assume Vi=1sin(wt)

$$V_o = \sqrt{1^2 + (1000\omega)^2}\sin\left(\omega t + \text{atan}\left(\frac{1}{1000\omega}\right)\right)$$

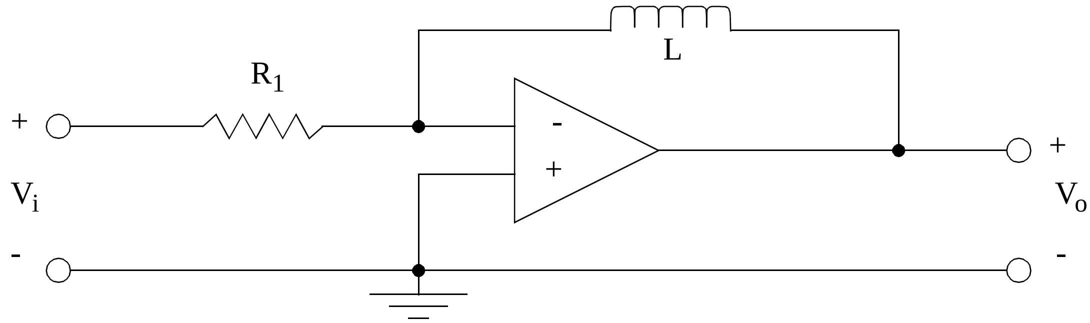
d)

7.11 ASSIGNMENT PROBLEMS

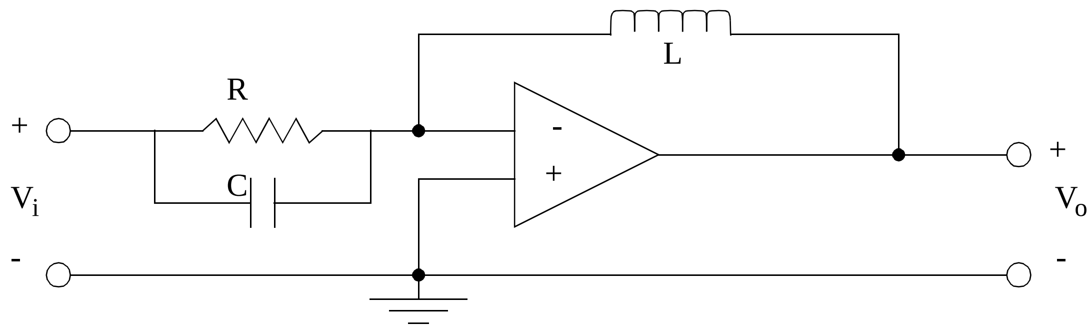
1. Write the differential equation for the following schematic.



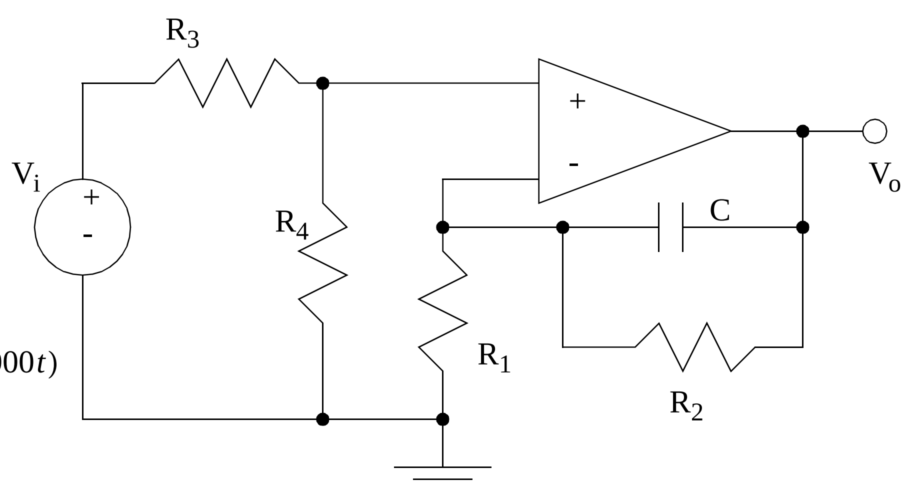
2. Write the differential equation for the following schematic.



3. Write the differential equation for the following schematic.



4. Develop the differential equation(s) for the system below, and use them to find the response to the following inputs. Assume that the circuit is off initially.



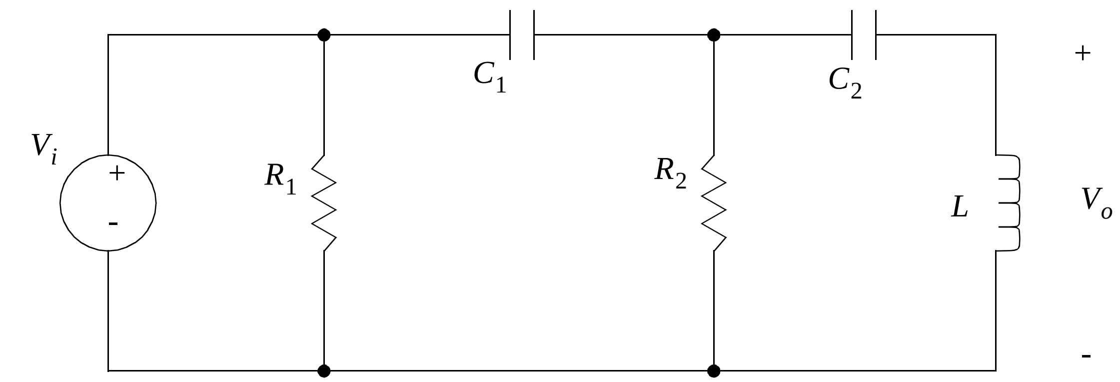
$R_1 = R_2 = R_3 = R_4 = 1\text{Kohm}$
C=1uF

a) $V_i = 5\sin(100t)$

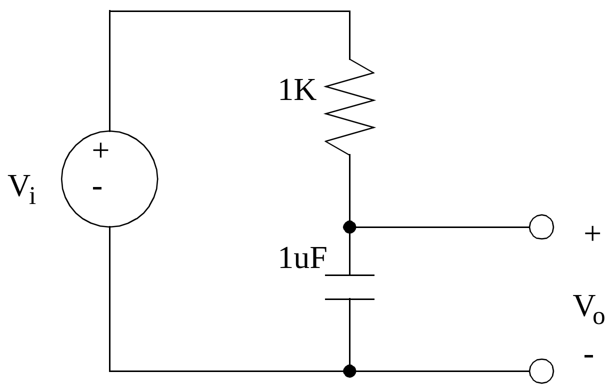
b) $V_i = 5\sin(1000000t)$

c) $V_i = 5$

5. Write the input-output equation for the circuit below.



6. Study the circuit below. Assume that for t<0s the circuit is discharged and off. Starting at t=0s an input of Vi=5sin(100,000t) is applied.



a) Write a differential equation and then a transfer function describing the circuit.
b) Find the output of the circuit using explicit integration (i.e., homogeneous and particular solutions).