31. VECTORS

Topics:

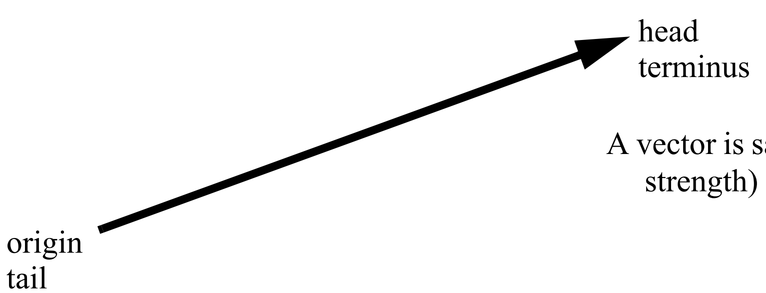
-

Objectives:

-

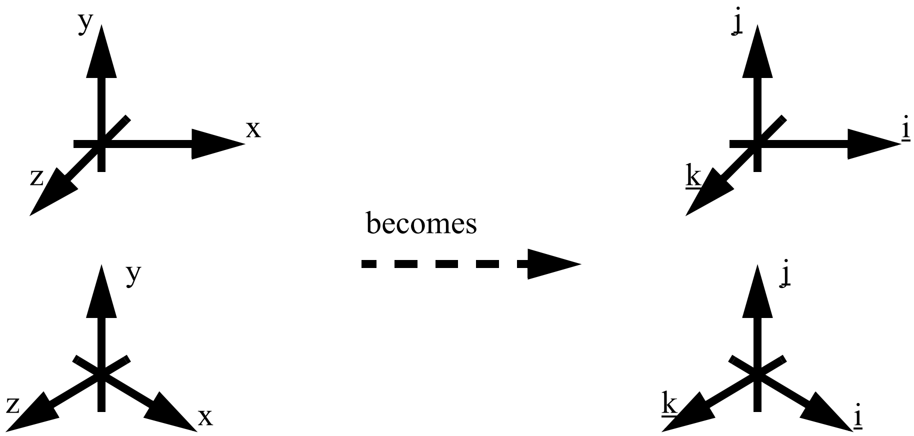
31.1 Introduction

• Vectors are often drawn with arrows, as shown below,



A vector is said to have magnitude (length or strength) and direction.

• Cartesian notation is also a common form of usage.

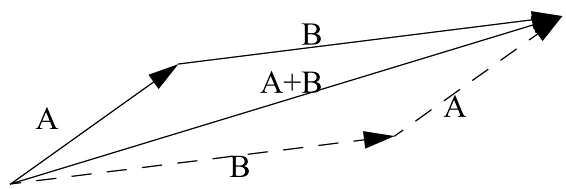


• Vectors can be added and subtracted, numerically and graphically,

$$A = (2, 3, 4) \qquad A + B = (2 + 7, 3 + 8, 4 + 9)$$

$$B = (7, 8, 9) \qquad A - B = (2 - 7, 3 - 8, 4 - 9)$$

Parallelogram Law



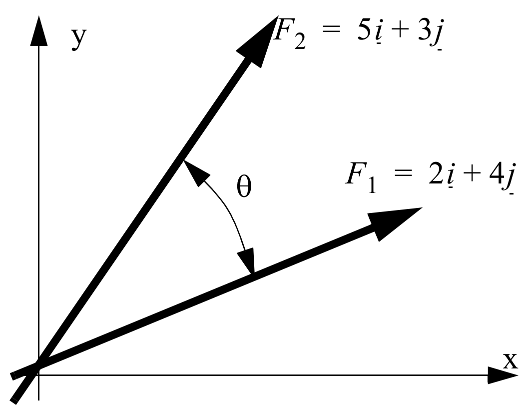
31.1.1 Dot (Scalar) Product

• We can use a dot product to find the angle between two vectors

$$\cos\theta = \frac{F_1 \bullet F_2}{|F_1||F_2|}$$

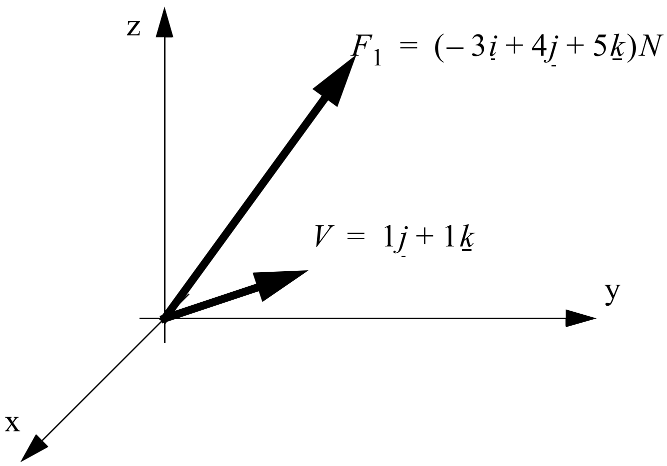
$$\therefore \theta = \operatorname{acos}\left(\frac{(2)(5) + (4)(3)}{\sqrt{2^2 + 4^2}\sqrt{5^2 + 3^2}}\right)$$

$$\therefore \theta = \operatorname{acos}\left(\frac{22}{(4.47)(6)}\right) = 32.5°$$



• We can use a dot product to project one vector onto another vector.

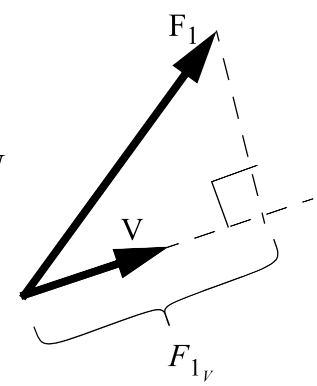
We want to find the component of force F_1 that projects onto the vector V. To do this we first convert V to a unit vector, if we do not, the component we find will be multiplied by the magnitude of V.



$$\lambda_V = \frac{V}{|V|} = \frac{1\underline{j} + 1\underline{k}}{\sqrt{1^2 + 1^2}} = 0.707\underline{j} + 0.707\underline{k}$$

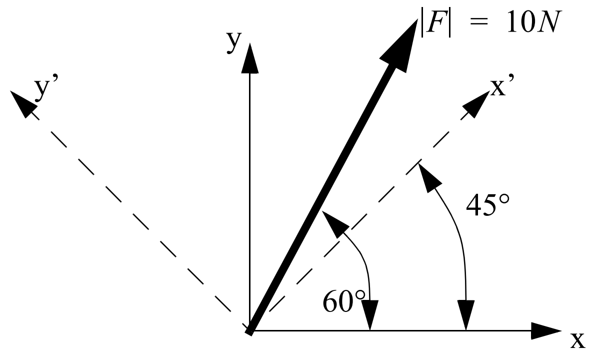
$$F_{1_V} = \lambda_V \bullet F_1 = (0.707\underline{j} + 0.707\underline{k}) \bullet (-3\underline{i} + 4\underline{j} + 5\underline{k})N$$

$$\therefore F_{1_V} = (0)(-3) + (0.707)(4) + (0.707)(5) = 6N$$



• We can consider the basic properties of the dot product and units vectors.

Unit vectors are useful when breaking up vector magnitudes and direction. As an example consider the vector, and the displaced x-y axes shown below as x'-y'.



We could write out 5 vectors here, relative to the x-y axis,

$$\text{x axis} = 2\underset{\sim}{i}$$
$$\text{y axis} = 3\underset{\sim}{j}$$
$$\text{x' axis} = 1\underset{\sim}{i} + 1\underset{\sim}{j}$$
$$\text{y' axis} = -1\underset{\sim}{i} + 1\underset{\sim}{j}$$
$$F = 10N\angle 60° = (10\cos 60°)\underset{\sim}{i} + (10\sin 60°)\underset{\sim}{j}$$

None of these vectors has a magnitude of 1, and hence they are not unit vectors. But, if we find the equivalent vectors with a magnitude of one we can simplify many tasks. In particular if we want to find the x and y components of F relative to the x-y axis we can use the dot product.

$$\lambda_x = 1\underset{\sim}{i} + 0\underset{\sim}{j} \quad \text{(unit vector for the x-axis)}$$

$$F_x = \lambda_x \bullet F = (1\underset{\sim}{i} + 0\underset{\sim}{j}) \bullet [(10\cos 60°)\underset{\sim}{i} + (10\sin 60°)\underset{\sim}{j}]$$
$$\therefore \quad = (1)(10\cos 60°) + (0)(10\sin 60°) = 10N\cos 60°$$

This result is obvious, but consider the other obvious case where we want to project a vector onto itself,

$$\lambda_F = \frac{F}{|F|} = \frac{10\cos 60° \underset{\sim}{i} + 10\sin 60° \underset{\sim}{j}}{10} = \cos 60° \underset{\sim}{i} + \sin 60° \underset{\sim}{j}$$

Incorrect - Not using a unit vector

$$F_F = F \bullet F$$
$$= ((10\cos 60°)\underset{\sim}{i} + (10\sin 60°)\underset{\sim}{j}) \bullet ((10\cos 60°)\underset{\sim}{i} + (10\sin 60°)\underset{\sim}{j})$$
$$= (10\cos 60°)(10\cos 60°) + (10\sin 60°)(10\sin 60°)$$
$$= 100((\cos 60°)^2 + (\sin 60°)^2) = \cancel{100}$$

Using a unit vector

$$F_F = F \bullet \lambda_F$$
$$= ((10\cos 60°)\underset{\sim}{i} + (10\sin 60°)\underset{\sim}{j}) \bullet ((\cos 60°)\underset{\sim}{i} + (\sin 60°)\underset{\sim}{j})$$
$$= (10\cos 60°)(\cos 60°) + (10\sin 60°)(\sin 60°)$$
$$= 10((\cos 60°)^2 + (\sin 60°)^2) = 10$$

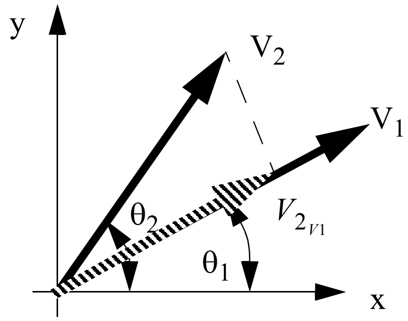
Correct

Now consider the case where we find the component of F in the x' direction. Again, this can be done using the dot product to project F onto a unit vector.

$$u_{x'} = \cos 45° \underset{\sim}{i} + \sin 45° \underset{\sim}{j}$$

$$F_{x'} = F \bullet \lambda_{x'} = ((10\cos 60°)\underset{\sim}{i} + (10\sin 60°)\underset{\sim}{j}) \bullet ((\cos 45°)\underset{\sim}{i} + (\sin 45°)\underset{\sim}{j})$$
$$= (10\cos 60°)(\cos 45°) + (10\sin 60°)(\sin 45°)$$
$$= 10(\cos 60° \cos 45° + \sin 60° \sin 45°) = 10(\cos(60° - 45°))$$

Here we see a few cases where the dot product has been applied to find the vector projected onto a unit vector. Now finally consider the more general case,



First, by inspection, we can see that the component of V_2 (projected) in the direction of V_1 will be,

$$\left|V_{2_{V1}}\right| = \left|V_2\right|\cos(\theta_2 - \theta_1)$$

Next, we can manipulate this expression into the dot product form,

$$= \left|V_2\right|(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$

$$= \left|V_2\right|[(\cos\theta_1 \underset{\sim}{i} + \sin\theta_1 \underset{\sim}{j}) \bullet (\cos\theta_2 \underset{\sim}{i} + \sin\theta_2 \underset{\sim}{j})]$$

$$= \left|V_2\right|\left[\frac{V_1}{\left|V_1\right|} \bullet \frac{V_2}{\left|V_2\right|}\right] = \left|V_2\right|\left[\frac{V_1 \bullet V_2}{\left|V_1\right|\left|V_2\right|}\right] = \frac{V_1 \bullet V_2}{\left|V_1\right|} = V_2 \bullet \lambda_{V_1}$$

Or more generally,

$$\left|V_{2_{V1}}\right| = \left|V_2\right|\cos(\theta_2 - \theta_1) = \left|V_2\right|\left[\frac{V_1 \bullet V_2}{\left|V_1\right|\left|V_2\right|}\right]$$

$$\therefore \left|V_2\right|\cos(\theta_2 - \theta_1) = \left|V_2\right|\left[\frac{V_1 \bullet V_2}{\left|V_1\right|\left|V_2\right|}\right]$$

$$\therefore \cos(\theta_2 - \theta_1) = \left[\frac{V_1 \bullet V_2}{\left|V_1\right|\left|V_2\right|}\right]$$

*Note that the dot product also works in 3D, and similar proofs are used.

• In Scilab,

```
A = [1 2 3];
B = [4 5 6];
dot = A'*B;
```

31.1.2 Cross Product

• First, consider an example,

$$F = (-6.43\underline{i} + 7.66\underline{j} + 0\underline{k})N$$

$$d = (2\underline{i} + 0\underline{j} + 0\underline{k})m$$

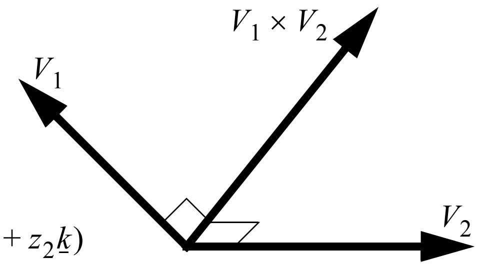
$$M = d \times F = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2m & 0m & 0m \\ -6.43N & 7.66N & 0N \end{bmatrix}$$

NOTE: note that the cross product here is for the right hand rule coordinates. If the left handed coordinate system is used F and d should be reversed.

$$\therefore M = (0m0N - 0m(7.66N))\underline{i} - (2m0N - 0m(-6.43N))\underline{j} + (2m(7.66N) - 0m(-6.43N))\underline{k} = 15.3\underline{k}(\mathrm{mN})$$

NOTE: there are two things to note about the solution. First, it is a vector. Here there is only a z component because this vector points out of the page, and a rotation about this vector would rotate on the plane of the page. Second, this result is positive, because the positive sense is defined by the vector system. In this right handed system find the positive rotation by pointing your right hand thumb towards the positive axis (the 'k' means that the vector is about the z-axis here), and curl your fingers, that is the positive direction.

• The basic properties of the cross product are, The cross (or vector) product of two vectors will yield a new vector perpendicular to both vectors, with a magnitude that is a product of the two magnitudes.



$$V_1 \times V_2 = (x_1\underline{i} + y_1\underline{j} + z_1\underline{k}) \times (x_2\underline{i} + y_2\underline{j} + z_2\underline{k})$$

$$V_1 \times V_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

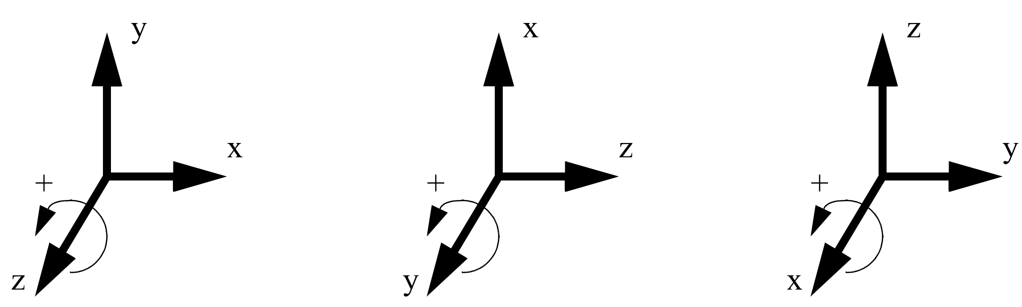
$$V_1 \times V_2 = (y_1z_2 - z_1y_2)\underline{i} + (z_1x_2 - x_1z_2)\underline{j} + (x_1y_2 - y_1x_2)\underline{k}$$

We can also find a unit vector normal 'n' to the vectors 'V1' and 'V2' using a cross product, divided by the magnitude.

$$\lambda_n = \frac{V_1 \times V_2}{|V_1 \times V_2|}$$

• When using a left/right handed coordinate system,

The positive orientation of angles and moments about an axis can be determined by pointing the thumb of the right hand along the axis of rotation. The fingers curl in the positive direction.



• The properties of the cross products are, The cross product is distributive, but not associative. This allows us to collect terms in a cross product operation, but we cannot change the order of the cross product.

$r_1 \times F + r_2 \times F = (r_1 + r_2) \times F$	DISTRIBUTIVE
$r \times F \neq F \times r$ but $r \times F = -(F \times r)$	NOT ASSOCIATIVE

• In Scilab,

```
function val=crossproduct(A, B) // No function is defined so use the following
        val = [A(2) * B(3) - A(3) * B(2) ;
        A(3) * B(1) - A(1) * B(3)
        A(1) * B(2) - A(2) * B(1)];
endfunction
```

31.2 Problems

1.