30. TRIGONOMETRY

Topics:

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Objectives:

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30.1 Introduction

• Angles degrees and radians

$$360° = 2\pi radians$$

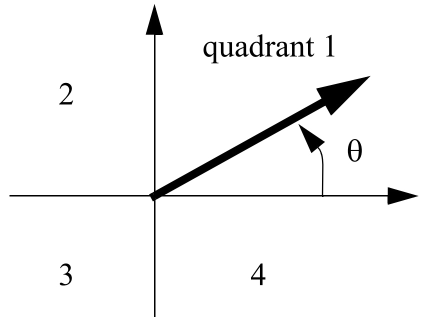
$$380° = 20° = -340°$$

$$60\text{minutes} = 60' = 1°$$

$$60\text{seconds} = 60'' = 1'$$

• Most computers do calculations in radians

• Angle quadrants,



30.1.1 Functions

• The basic trigonometry functions are,

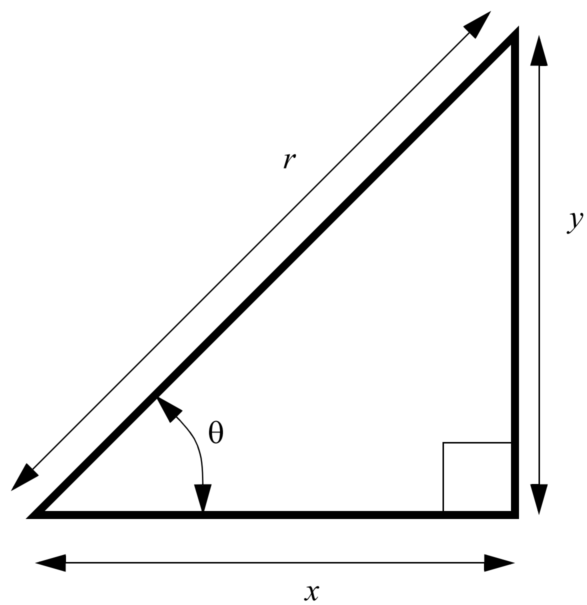
$$\sin\theta = \frac{y}{r} = \frac{1}{\csc\theta}$$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sec\theta}$$

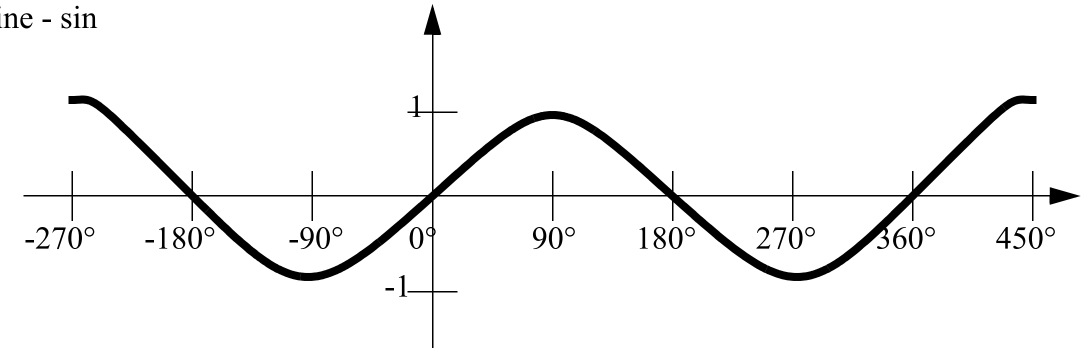
$$\tan\theta = \frac{y}{x} = \frac{1}{\cot\theta} = \frac{\sin\theta}{\cos\theta}$$

Pythagorean Formula:

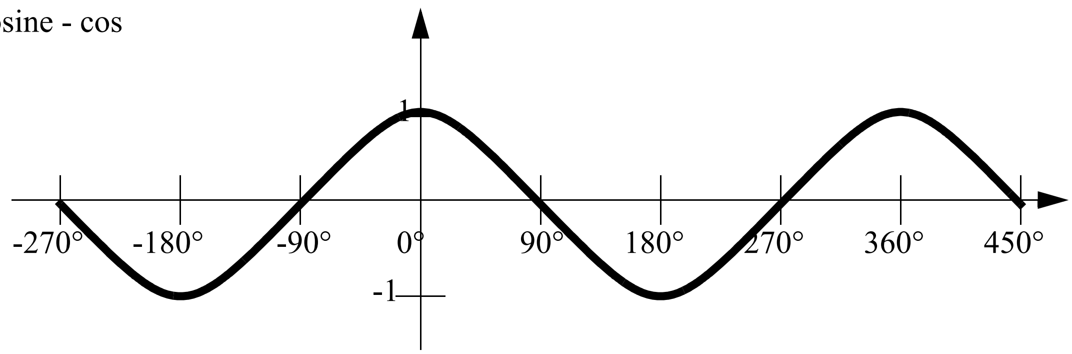
$$r^2 = x^2 + y^2$$



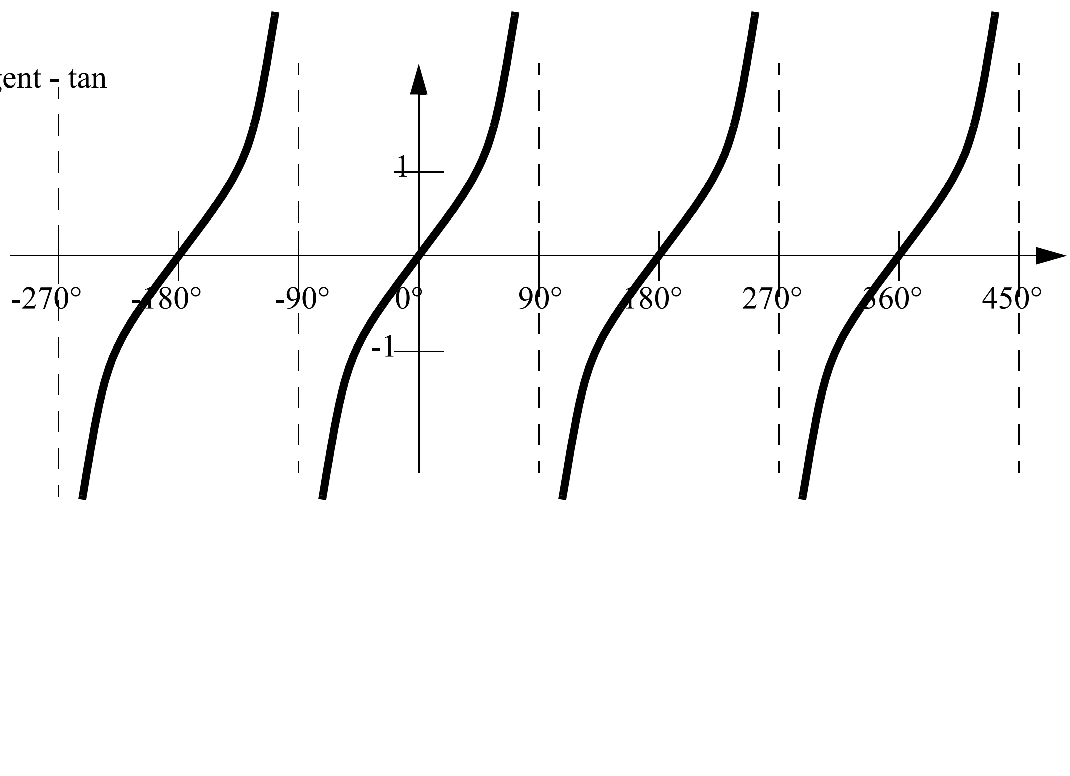
• Graphs of these functions are given below, Sine - sin

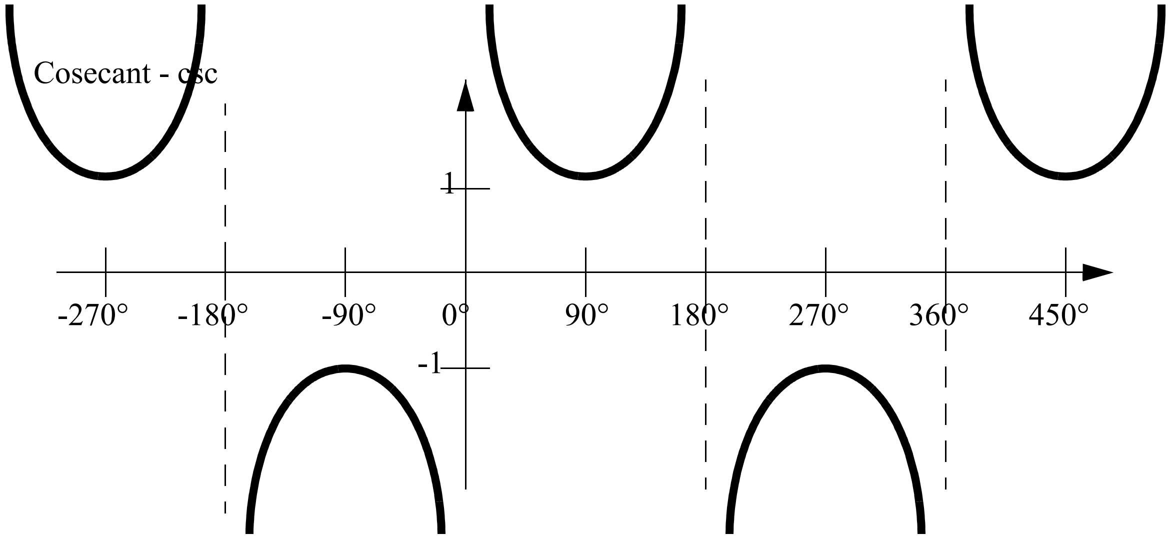


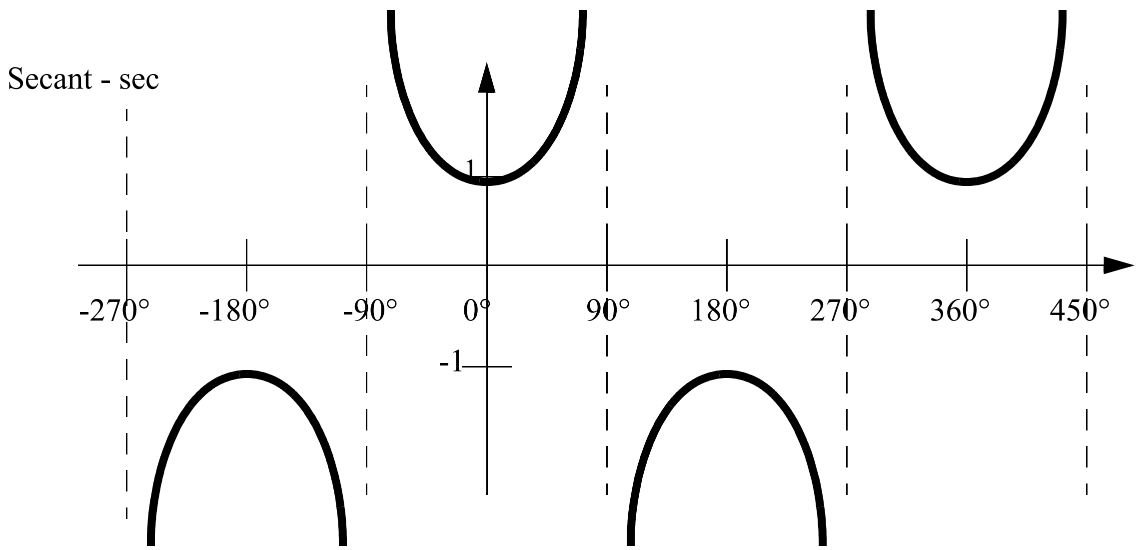
Cosine - cos



Tangent - tan




Cosecant - csc
1
-1
-270°
-180°
-90°
0°
90°
180°
270°
360°
450°


Secant - sec
1
-1
-270°
-180°
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0°
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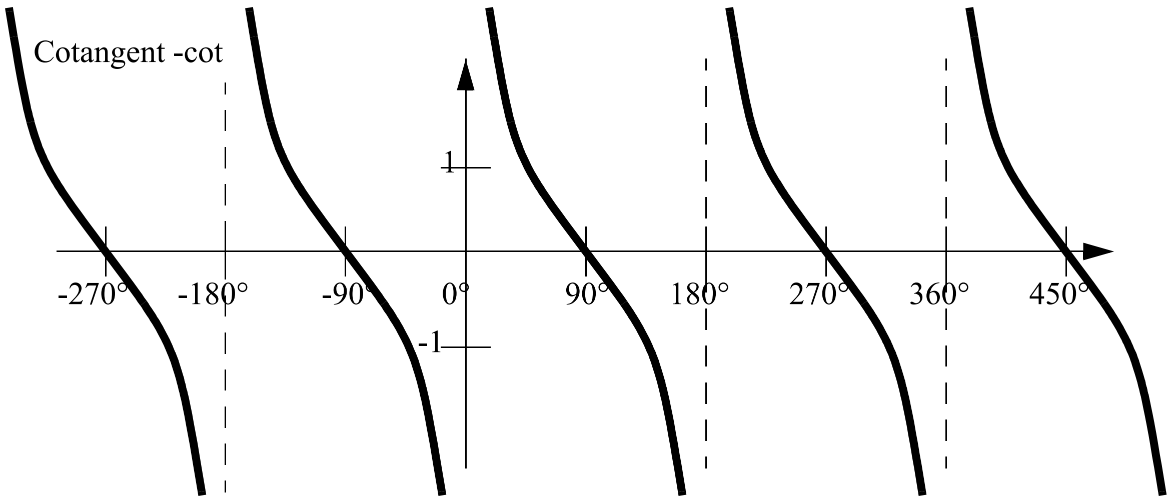
360°
450°



30.1.2 Inverse Functions

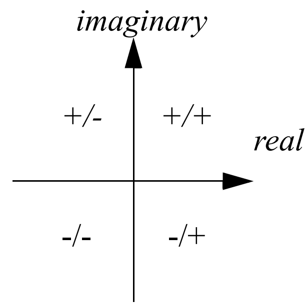
• Inverse Functions

$$\tan^{-1}\left(\frac{y}{x}\right) = \text{atan}\left(\frac{y}{x}\right) = \theta$$

$$\sin^{-1}\left(\frac{y}{r}\right) = \text{asin}\left(\frac{y}{r}\right) = \theta$$

$$\cos^{-1}\left(\frac{x}{r}\right) = \text{acos}\left(\frac{x}{r}\right) = \theta$$

Note: recall that $\tan\theta = \frac{Re}{Im}$ *but the* $\operatorname{atan}\theta$ *function in calculators and software only returns values between -90 to 90 degrees. To compensate for this the sign of the real and imaginary components must be considered to determine where the angle lies. If it lies beyond the -90 to 90 degree range the correct angle can be obtained by adding or subtracting 180 degrees.*



• Note: trig calculations can take a while and should be minimized or avoided in programs.

• Scilab example,

```
sin(3.14159)
asin(0.5)
cos(3.14159)
acos(0.5)
tan(3.14159)
atan(0.5)
atan(1.0, 0.5)
```

30.1.3 Triangles

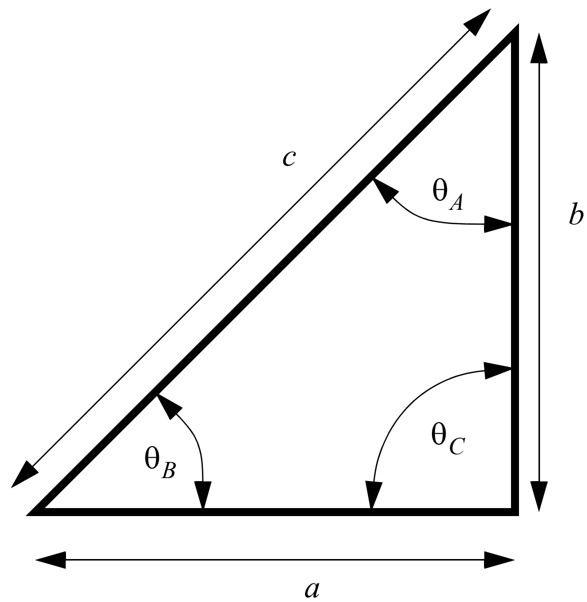
• NOTE: Keep in mind when finding these trig values, that any value that does not lie in the right hand quadrants of cartesian space, may need additions of ±90° or ±180°.

Cosine Law:

$$c^2 = a^2 + b^2 - 2ab\cos\theta_c$$

Sine Law:

$$\frac{a}{\sin\theta_A} = \frac{b}{\sin\theta_B} = \frac{c}{\sin\theta_C}$$



30.1.4 Relationships

• Now a group of trigonometric relationships will be given. These are often best used when attempting to manipulate equations.

$$\sin(\theta) = \sin(\theta \pm 360n) \qquad n \in I$$

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\sin\theta = \cos(\theta - 90°) = \cos(90° - \theta) = \text{etc.}$$

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2 \qquad \text{OR} \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2 \qquad \text{OR} \qquad \cos(2\theta) = (\cos\theta)^2 - (\sin\theta)^2$$

$$\tan(\theta_1 \pm \theta_2) = \frac{\tan\theta_1 \pm \tan\theta_2}{1 \mp \tan\theta_1\tan\theta_2} \qquad 1 + (\tan\theta)^2 = (\sec\theta)^2$$

$$\cot(\theta_1 \pm \theta_2) = \frac{\cot\theta_1\cot\theta_2 \mp 1}{\tan\theta_2 \pm \tan\theta_1} \qquad 1 + (\cot\theta)^2 = (\csc\theta)^2$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$$

-ve if in left hand quadrants

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta} = \frac{1 - \cos\theta}{\sin\theta}$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

• Scilab for trig identities,

EXAMPLES OF TRIG IDENTITIES

• These can also be related to complex exponents,

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

30.1.5 Hyperbolic Functions

• The basic definitions are given below,

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} = \text{hyperbolic sine of x}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} = \text{hyperbolic cosine of x}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \text{hyperbolic tangent of x}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^{x} - e^{-x}} = \text{hyperbolic cosecant of x}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^{x} + e^{-x}} = \text{hyperbolic secant of x}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}} = \text{hyperbolic cotangent of x}$$

• some of the basic relationships are,

$$\sinh(-x) = -\sinh(x)$$

$$\cosh(-x) = \cosh(x)$$

$$\tanh(-x) = -\tanh(x)$$

$$\operatorname{csch}(-x) = -\operatorname{csch}(x)$$

$$\operatorname{sech}(-x) = \operatorname{sech}(x)$$

$$\coth(-x) = -\coth(x)$$

• Some of the more advanced relationships are,

$$(\cosh x)^2 - (\sinh x)^2 = (\operatorname{sech} x)^2 + (\tanh x)^2 = (\coth x)^2 - (\operatorname{csch} x)^2 = 1$$

$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$$

$$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$$

$$\tanh(x \pm y) = \frac{\tanh(x) \pm \tanh(y)}{1 \pm \tanh(x)\tanh(y)}$$

• Some of the relationships between the hyperbolic, and normal trigonometry functions are,

$$\sin(jx) = j\sinh(x) \qquad j\sin(x) = \sinh(jx)$$

$$\cos(jx) = \cosh(x) \qquad \cos(x) = \cosh(jx)$$

$$\tan(jx) = j\tanh(x) \qquad j\tan(x) = \tanh(jx)$$

30.1.6 Special Relationships

• The Small Angle Approximation

$$\sin\theta = \theta \qquad \text{when} \qquad \theta \approx 0$$

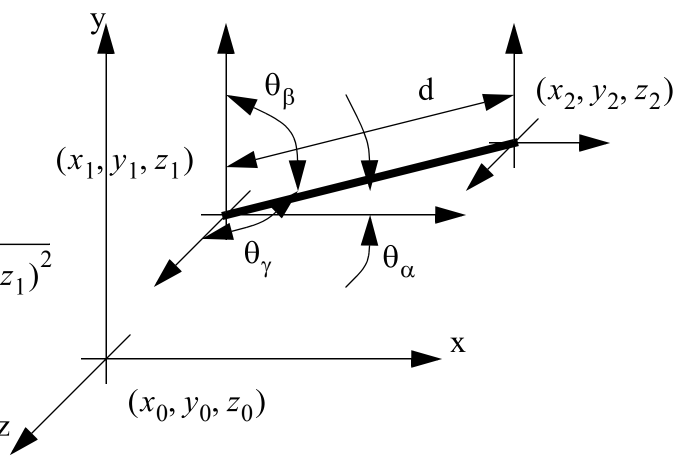
30.1.7 Planes, Lines, etc.

• The most fundamental mathematical geometry is a line. The basic relationships are given below,

$y = mx + b$	defined with a slope and intercept
$m_{\text{perpendicular}} = \frac{1}{m}$	a slope perpendicular to a line
$m = \frac{y_2 - y_1}{x_2 - x_1}$	the slope using two points
$\frac{x}{a} + \frac{y}{b} = 1$	as defined by two intercepts

• If we assume a line is between two points in space, and that at one end we have a local reference frame, there are some basic relationships that can be derived.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



The direction cosines of the angles are,

$$\theta_\alpha = \operatorname{acos}\left(\frac{x_2 - x_1}{d}\right) \qquad \theta_\beta = \operatorname{acos}\left(\frac{y_2 - y_1}{d}\right) \qquad \theta_\gamma = \operatorname{acos}\left(\frac{z_2 - z_1}{d}\right)$$

$$(\cos\theta_\alpha)^2 + (\cos\theta_\beta)^2 + (\cos\theta_\gamma)^2 = 1$$

The equation of the line is,

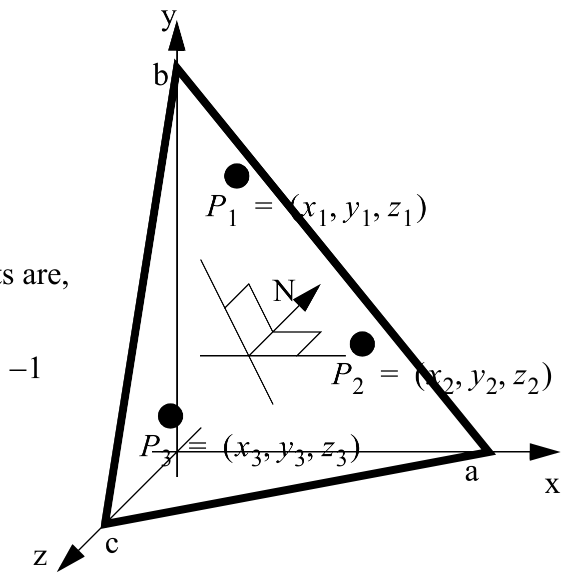
$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$ Explicit

$$(x, y, z) = (x_1, y_1, z_1) + t((x_2, y_2, z_2) - (x_1, y_1, z_1))$$ Parametric t=[0,1]

• The relationships for a plane are,

The explicit equation for a plane is,

$$Ax + By + Cy + D = 0$$



where the coefficients defined by the intercepts are,

$$A = \frac{1}{a} \qquad B = \frac{1}{b} \qquad C = \frac{1}{c} \qquad D = -1$$

The determinant can also be used,

$$det\begin{bmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ x-x_3 & y-y_3 & z-z_3 \end{bmatrix} = 0$$

$$\therefore det\begin{bmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_1 & z_3-z_1 \end{bmatrix}(x-x_1) + det\begin{bmatrix} z_2-z_1 & x_2-x_1 \\ z_3-z_1 & x_3-x_1 \end{bmatrix}(y-y_1)$$

$$+ det\begin{bmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{bmatrix}(z-z_1) = 0$$

The normal to the plane (through the origin) is,

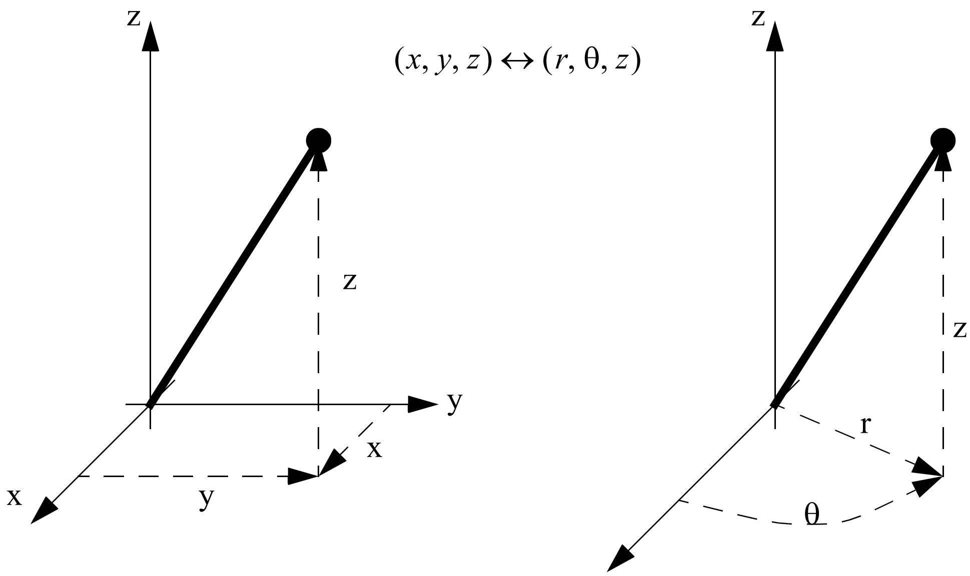
$$(x, y, z) = t(A, B, C)$$

30.2 Coordinate Systems

30.2.1 Cylindrical Coordinates

• Basically, these coordinates appear as if the cartesian box has been replaced with a cylinder,

$$(x, y, z) \leftrightarrow (r, \theta, z)$$



$$x = r\cos\theta$$

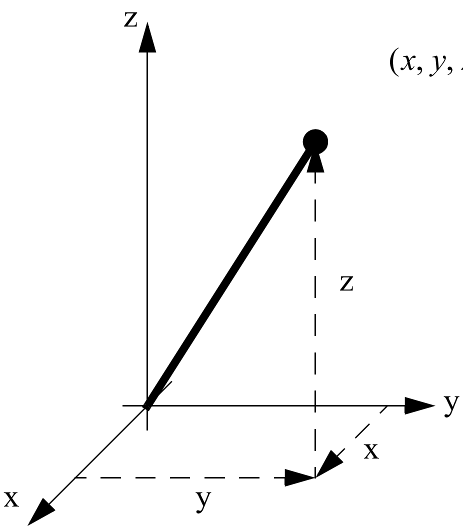
$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$

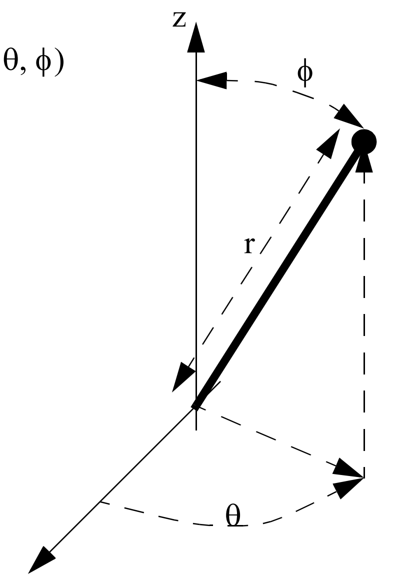
$$\theta = \operatorname{atan}\left(\frac{y}{x}\right)$$

30.2.2 Spherical Coordinates

• This system replaces the cartesian box with a sphere,



$$(x, y, z) \leftrightarrow (r, \theta, \phi)$$



$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \operatorname{atan}\left(\frac{y}{x}\right)$$

$$\phi = \operatorname{acos}\left(\frac{z}{r}\right)$$

30.3 Problems

1. For the following angles, i) indicate the quadrants, ii) write the sine, cosine, and tangent values, iii) write an expression for all equivalent angles.

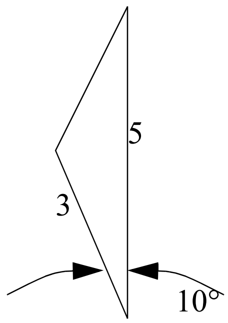
   a) $20°$

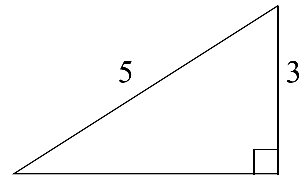
   b) $2rad$

   c) $-2rad$

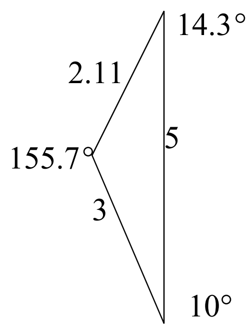
   d) $2000°$

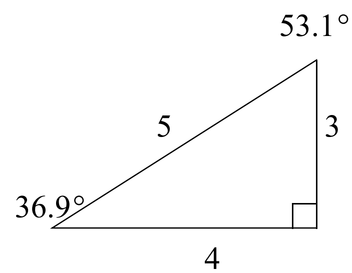
2. Find all of the missing side lengths and corner angles on the two triangles below,





ans.





3. A line that passes through the point (1, 2) and has a slope of 2. Find the equation for the line, and for a line perpendicular to it passing through the given point. (ans. y = 2x + a, y = -0.5x+b)

4. Convert the following angles to/from degrees or radians.

to rad.: 30°, 280°, 192°5'30"

to deg.: $\frac{5\pi}{36}rad, \frac{7\pi}{12}rad$

ans. $\frac{\pi}{6}rad, \frac{14}{9}\pi rad, 1.067\pi rad$

25°, 105°

5. On a circle with a diameter of 0.5in. what is,

a) the arc length for a 1 rad angle
b) the arc length for a 20 degree angle
c) the circumference.
d) the angle resulting in a 0.5m arc.

(ans. 0.25in, PI/36 in, 0.5 PI in., 78.7rad.)

6. If a 30cm radius rotating mass has a rotational rate of 200rpm, how fast is a point at the outside moving? What angular velocity is required for an outside speed of 2m/s? (ans. 6.28m/s, 6.67 rad/s)

7.Given the (x, y) points below, find the angle

a)	$(3, 4)$	(ans. $53°$
b)	$(-3, 4)$	$126°$
c)	$(3, -4)$	$-53°$
d)	$(-3, -4)$	$-126°$

8.Convert the following to a function with the smallest positive angle possible

a)	$\sin(200°)$	(ans. $-\sin(20°)$
b)	$\tan(-170°)$	$\tan(10°)$
c)	$\cos(325°)$	$\cos(35°)$
d)	$\cos(3.5\pi rad)$	$\cos(90°)$ or $\sin(0°)$

9.Given a triangle ABC find the missing side lengths or angles

θ_A	θ_A	θ_A	L_A	L_B	L_C

10. Simplify the following expressions.

$$\sin 2\theta\left(\frac{(\cos 2\theta)^2}{\sin 2\theta} + \sin 2\theta\right)$$ (ans. = 1)

11.Prove the following

a) $$\frac{\sin\theta\cos\theta}{\tan\theta} = \frac{(\cos\theta)^2 - (\sin\theta)^2}{1 - (\tan\theta)^2}$$

b) $$\sin\theta = 1 - \frac{(\cos\theta)^2}{1 + (\sin\theta)}$$

c) $$(1 + (\tan\theta)^2)(1 - (\sin\theta)^2) = 1$$

d) $$\sin(\theta_1 + \theta_2) - \sin(\theta_1 - \theta_2) = 2\cos\theta_1\sin\theta_2$$

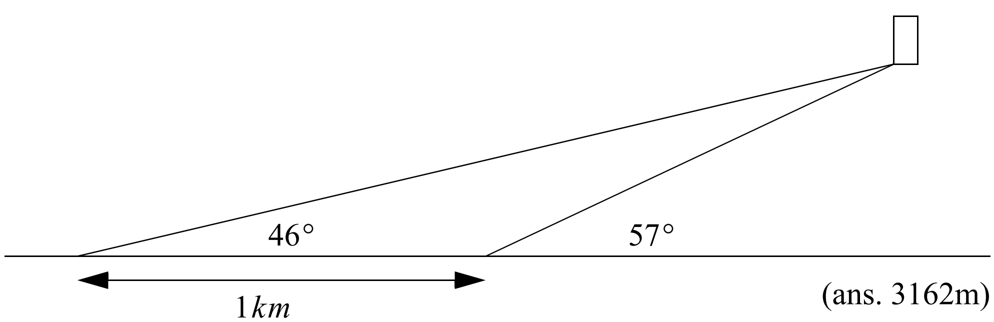
e) $$\sin(\theta_1 + \theta_2)(-\sin(\theta_1 - \theta_2)) = (\sin(\theta_1))^2 - (\sin(\theta_2))^2$$

f) $$\tan\theta\sin 2\theta = 2(\sin\theta)^2$$

12. Solve the equation,

$$(\cos\theta)^2 + 2\sin\theta - 2 = 0$$ (ans. $\theta =$

13. Two observers measure the height of a rocket by angle. Calculate the height of the rocket.



(ans. 3162m)