4. TRANSFORMS

***** This contains additions and sections by Dr. Andrew Sterian.

Topics:

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Objectives:

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4.1 Laplace Transforms

• The Laplace transform allows us to reverse time. And, as you recall from before the inverse of time is frequency. Because we are normally concerned with response, the Laplace transform is much more useful in system analysis.

• The basic Laplace transform equations is shown below,

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

where,

$f(t)$ = the function in terms of time t

$F(s)$ = the function in terms of the Laplace s

4.1.1 Laplace Transform Tables

• Basic Laplace Transforms for operational transformations are given below,

TIME DOMAIN	FREQUENCY DOMAIN
$Kf(t)$	$Kf(s)$
$f_1(t) + f_2(t) - f_3(t) + \ldots$	$f_1(s) + f_2(s) - f_3(s) + \ldots$
$\frac{df(t)}{dt}$	$sf(s) - f(0^-)$
$\frac{d^2 f(t)}{dt^2}$	$s^2 f(s) - sf(0^-) - \frac{df(0^-)}{dt}$
$\frac{d^n f(t)}{dt^n}$	$s^n f(s) - s^{n-1} f(0^-) - s^{n-2} \frac{df(0^-)}{dt} - \ldots - \frac{d^n f(0^-)}{dt^n}$
$\int_0^t f(t) dt$	$\frac{f(s)}{s}$
$f(t-a)u(t-a), a > 0$	$e^{-as} f(s)$
$e^{-at} f(t)$	$f(s-a)$
$f(at), a > 0$	$\frac{1}{a} f\left(\frac{s}{a}\right)$
$tf(t)$	$\frac{-df(s)}{ds}$
$t^n f(t)$	$(-1)^n \frac{d^n f(s)}{ds^n}$
$\frac{f(t)}{t}$	$\int_s^{\infty} f(u) du$

• A set of useful functional Laplace transforms are given below,

TIME DOMAIN	FREQUENCY DOMAIN
A	$\frac{A}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
Ae^{-at}	$\frac{A}{s-a}$
Ate^{-at}	$\frac{A}{(s-a)^2}$
$2\|A\|e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{s+\alpha-\beta j}+\frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}$
$2t\|A\|e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{(s+\alpha-\beta j)^2}+\frac{A^{\text{complex conjugate}}}{(s+\alpha+\beta j)^2}$

• Laplace transforms can be used to solve differential equations.

4.2 z-Transforms

• For a discrete-time signal $x[n]$, the two-sided z-transform is defined by $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$.

The one-sided z-transform is defined by $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$. In both cases, the z-transform is a polynomial in the complex variable z.

• The inverse z-transform is obtained by contour integration in the complex plane $x[n] = \frac{1}{j2\pi}\oint X(z)z^{n-1}dz$. This is usually avoided by partial fraction inversion techniques, similar to the Laplace transform.

• Along with a z-transform we associate its region of convergence (or ROC). These are the values of z for which $X(z)$ is bounded (i.e., of finite magnitude).

• Some common z-transforms are shown below.

Table 1: Common z-transforms

Signal $x[n]$	z-Transform $X(z)$	ROC
$\delta[n]$	1	All z
$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$nu[n]$	$\dfrac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
$n^2u[n]$	$\dfrac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\lvert z\rvert > 1$
$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$(-a^n)u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$(-na^n)u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\cos(\omega_0 n)u[n]$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$a^n\cos(\omega_0 n)u[n]$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Table 1: Common z-transforms

Signal $x[n]$	z-Transform $X(z)$	ROC
$a^n \sin(\omega_0 n) u[n]$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0 + a^2 z^{-2}}$	$\|z\| > \|a\|$
$\dfrac{n!}{k!(n-k)!} u[n]$	$\dfrac{z^{-k}}{(1-z^{-1})^{k+1}}$	$\|z\| > 1$

• The z-transform also has various properties that are useful. The table below lists properties for the two-sided z-transform. The one-sided z-transform properties can be derived from the ones below by considering the signal $x[n]u[n]$ instead of simply $x[n]$.

Table 2: Two-sided z-Transform Properties

Property	Time Domain	z-Domain	ROC
Notation	$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$	$r_2 < \|z\| < r_1$ ROC_1 ROC_2
Linearity	$\alpha x_1[n] + \beta x_2[n]$	$\alpha X_1(z) + \beta X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time Shifting	$x[n-k]$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
z-Domain Scaling	$a^n x[n]$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time Reversal	$x[-n]$	$X(z^{-1})$	$\dfrac{1}{r_1} < \|z\| < \dfrac{1}{r_2}$
z-Domain Differentiation	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$

Table 2: Two-sided z-Transform Properties

Property	Time Domain	z-Domain	ROC
Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of ROC_1 and ROC_2
Multiplication	$x_1[n]x_2[n]$	$\frac{1}{j2\pi}\oint X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$	At least $r_{1l}r_{2l} < \lvert z\rvert < r_{1u}r_{2u}$
Initial value theorem	$x[n]$ causal	$x[0] = \lim_{z \to \infty} X(z)$	

4.3 Fourier Series

- These series describe functions by their frequency spectrum content. For example a square wave can be approximated with a sum of a series of sine waves with varying magnitudes.

- The basic definition of the Fourier series is given below.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

4.4 Problems

8a. Find y(t).

$$\frac{y(s)}{x(s)} = \frac{s^2 + 4s}{s^2 + 6s + 9} \qquad x(t) = 5$$

4.5 Challenge Problems