32. MATRICES

Topics:

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Objectives:

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32.1 Introduction

- Matrices allow simple equations that drive a large number of repetitive calculations - as a result they are found in many computer applications.

- A matrix has the form seen below,

n columns

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix}$$

m rows

If n=m then the matrix is said to be square. Many applications require square matrices. We may also represent a matrix as a 1-by-3 for a vector.

• In Scilab,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

```
A = [ 1 2 3 ; 4 5 6 ; 7 8 9 ]; // ; means next row
A(2, 3) // , means next element
A(1:2, 2:3) // : means a range
A(:, 2)
A(2, :)
A(2, $) // means last row or column
A($, 2)
// other commands - please use help to explore
    these
size(A); // returns the rows, columns, etc of A
ones(A); // fills the matrix with 1s
zeros(A); // fills the matrix with 0s
eye(5, 5); // creates a 5x5 identity matrix
diag(A); // gets the diagonal of matrix A
rand()
max()
rank()
cond()
spec()
trace()
// also try
B = [ 1 2 3 ];
C = [1 ; 2 ; 3 ];
D = 1:0.1:3;
E = [ B 4 ]; // adds a column to B to make E
F = [ C ; 4 ]; // add a row to C to make F
length(C); // the rows in C
G = A(2:3 , 1:2); // extracts a 2x2 from A
H = B. * C; // element wise operation, a 3x3 results
J = B^2; // same as B*B
K = B.^2; // each element is squared
```

32.1.1 Basic Matrix Operations

• Matrix operations are available for many of the basic algebraic expressions, examples are given below. There are also many restrictions - many of these are indicated.

$$A = 2 \qquad B = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{bmatrix} \qquad C = \begin{bmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{bmatrix} \qquad D = \begin{bmatrix} 21 \\ 22 \\ 23 \end{bmatrix} \qquad E = \begin{bmatrix} 24 & 25 & 26 \end{bmatrix}$$

Addition/Subtraction

$$A + B = \begin{bmatrix} 3+2 & 4+2 & 5+2 \\ 6+2 & 7+2 & 8+2 \\ 9+2 & 10+2 & 11+2 \end{bmatrix} \qquad B + C = \begin{bmatrix} 3+12 & 4+13 & 5+14 \\ 6+15 & 7+16 & 8+17 \\ 9+18 & 10+19 & 11+20 \end{bmatrix}$$

$$B + D = \text{not valid}$$

$$B - A = \begin{bmatrix} 3-2 & 4-2 & 5-2 \\ 6-2 & 7-2 & 8-2 \\ 9-2 & 10-2 & 11-2 \end{bmatrix} \qquad B + C = \begin{bmatrix} 3-12 & 4-13 & 5-14 \\ 6-15 & 7-16 & 8-17 \\ 9-18 & 10-19 & 11-20 \end{bmatrix}$$

$$B - D = \text{not valid}$$

$$A \cdot B = \begin{bmatrix} 3(2) & 4(2) & 5(2) \\ 6(2) & 7(2) & 8(2) \\ 9(2) & 10(2) & 11(2) \end{bmatrix} \qquad \frac{B}{A} = \begin{bmatrix} \frac{3}{2} & \frac{4}{2} & \frac{5}{2} \\ \frac{6}{2} & \frac{7}{2} & \frac{8}{2} \\ \frac{9}{2} & \frac{10}{2} & \frac{11}{2} \end{bmatrix}$$

$$B \cdot D = \begin{bmatrix} (3 \cdot 21 + 4 \cdot 22 + 5 \cdot 23) \\ (6 \cdot 21 + 7 \cdot 22 + 8 \cdot 23) \\ (9 \cdot 21 + 10 \cdot 22 + 11 \cdot 23) \end{bmatrix} \qquad D \cdot E = 21 \cdot 24 + 22 \cdot 25 + 23 \cdot 26$$

$$B \cdot C = \begin{bmatrix} (3 \cdot 12 + 4 \cdot 15 + 5 \cdot 18) & (3 \cdot 13 + 4 \cdot 16 + 5 \cdot 19) & (3 \cdot 14 + 4 \cdot 17 + 5 \cdot 20) \\ (6 \cdot 12 + 7 \cdot 15 + 8 \cdot 18) & (6 \cdot 13 + 7 \cdot 16 + 8 \cdot 19) & (6 \cdot 14 + 7 \cdot 17 + 8 \cdot 20) \\ (9 \cdot 12 + 10 \cdot 15 + 11 \cdot 18) & (9 \cdot 13 + 10 \cdot 16 + 11 \cdot 19) & (9 \cdot 14 + 10 \cdot 17 + 11 \cdot 20) \end{bmatrix}$$

$$\frac{B}{C}, \frac{B}{D}, \frac{D}{B}, \text{etc} = \text{not allowed (see inverse)}$$

Note: To multiply matrices, the first matrix must have the same number of columns as the second matrix has rows.

32.1.2 Determinants

- Determinants give a 'magnitude product' of a matrix. This can be though of as a general magnitude of the matrix.
- To find a determinant the matrix must must be square.
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- For a 2 by 2 matrix.

Determinant

$$|B| = 3\cdot\begin{vmatrix}7 & 8\\10 & 11\end{vmatrix} - 4\cdot\begin{vmatrix}6 & 8\\9 & 11\end{vmatrix} + 5\cdot\begin{vmatrix}6 & 7\\9 & 10\end{vmatrix} = 3\cdot(-3) - 4\cdot(-6) + 5\cdot(-3) = 0$$

$$\begin{vmatrix}7 & 8\\10 & 11\end{vmatrix} = (7\cdot 11) - (8\cdot 10) = -3$$

$$\begin{vmatrix}6 & 8\\9 & 11\end{vmatrix} = (6\cdot 11) - (8\cdot 9) = -6$$

$$\begin{vmatrix}6 & 7\\9 & 10\end{vmatrix} = (6\cdot 10) - (7\cdot 9) = -3$$

$|D|, |E|$ = not valid (matrices not square)

- For a 3 by 3 matrix

$$\begin{vmatrix}a & b & c\\d & e & f\\g & h & i\end{vmatrix} = a\begin{vmatrix}e & f\\h & i\end{vmatrix} - b\begin{vmatrix}d & f\\g & i\end{vmatrix} + c\begin{vmatrix}d & e\\g & h\end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

- Higher order matrices follow a similar pattern. For example a 4th order matrix has the pattern,

$$\begin{vmatrix}a & b & c & d\\e & f & g & h\\i & j & k & l\\m & n & o & p\end{vmatrix} = a\begin{vmatrix}f & g & h\\j & k & l\\n & o & p\end{vmatrix} - b\begin{vmatrix}e & g & h\\i & k & l\\m & o & p\end{vmatrix} + c\begin{vmatrix}e & f & h\\i & j & l\\m & n & p\end{vmatrix} - d\begin{vmatrix}e & f & g\\i & j & k\\m & n & o\end{vmatrix}$$

32.1.3 Transpose

$$B^T = \begin{bmatrix} 3 & 6 & 9 \\ 4 & 7 & 10 \\ 5 & 8 & 11 \end{bmatrix} \qquad D^T = \begin{bmatrix} 21 & 22 & 23 \end{bmatrix} \qquad E^T = \begin{bmatrix} 24 \\ 25 \\ 26 \end{bmatrix}$$

32.1.4 Adjoint Matrices

$$\|B\| = \begin{bmatrix} \begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} & -\begin{vmatrix} 6 & 8 \\ 10 & 11 \end{vmatrix} & \begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix} \\ -\begin{vmatrix} 4 & 5 \\ 10 & 11 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 9 & 11 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 9 & 10 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix} \end{bmatrix}^T$$

The matrix of determinant to the left is made up by getting rid of the row and column of the element, and then finding the determinant of what is left. Note the sign changes on alternating elements.

$\|D\|$ = invalid (must be square)

32.1.5 Inverse Matrices

$$D = B \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

To solve this equation for x,y,z we need to move B to the left hand side. To do this we use the inverse.

$$B^{-1}D = B^{-1} \cdot B \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B^{-1}D = I \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B^{-1} = \frac{||B||}{|B|}$$

In this case B is singular, so the inverse is undetermined, and the matrix is indeterminate.

$$D^{-1} = \text{invalid (must be square)}$$

• Some Scilab,

```
A = [ 1 2 3 ; 4 5 6 ; 7 8 9 ];
B = [ 10 ; 11 ; 12 ];
A' // transpose
det(A) // determinant
inv(A) // inverse
A^-1 // also inverse
spec(A)
[D, X] = bdiag(A)
linsolve(A, B)
A * B
B * A
A + B // will not work
```

32.1.6 Identity Matrix

This is a square matrix that is the matrix equivalent to ‘1’.

$$B \cdot I = I \cdot B = B$$

$$D \cdot I = I \cdot D = D$$

$$B^{-1} \cdot B = I$$

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{etc=I}$$

32.1.7 Eigenvalues

• The eigenvalue of a matrix is found using,

$$|A - \lambda I| = 0$$

32.1.8 Eigenvectors

32.2 Matrix Applications

32.2.1 Solving Linear Equations with Matrices

• Note: if the determinant of a matrix is 0, the matrix is singular and there is no solution for the linear equations

Given,

$$x + y = 5$$

$$2x + 3y = 8$$

The equations can be written in matrix form,

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

The solution is found using the inverse,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \frac{\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \frac{\begin{bmatrix} 15-8 \\ -10+8 \end{bmatrix}}{3-2} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

• In Scilab,

```
A = [1 1 ; 2 3];
B = [5; 8];
X = inv(A) * B
X = A \ B; // another way to solve linear equations
r = B - A * x; // The residual should be 0
```

• We can solve systems of equations using the inverse matrix,

Given,

$$2 \cdot x + 4 \cdot y + 3 \cdot z = 5$$

$$9 \cdot x + 6 \cdot y + 8 \cdot z = 7$$

$$11 \cdot x + 13 \cdot y + 10 \cdot z = 12$$

Write down the matrix, then rearrange, and solve.

$$\begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix} \qquad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

• We can solve systems of equations using Cramer's rule (with determinants),

Given,

$$2 \cdot x + 4 \cdot y + 3 \cdot z = 5$$

$$9 \cdot x + 6 \cdot y + 8 \cdot z = 7$$

$$11 \cdot x + 13 \cdot y + 10 \cdot z = 12$$

Write down the coefficient and parameter matrices,

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix}$$

Calculate the determinant for A (this will be reused), and calculate the determinants for matrices below. Note: when trying to find the first parameter 'x' we replace the first column in A with B.

$$x = \frac{\begin{vmatrix} 5 & 4 & 3 \\ 7 & 6 & 8 \\ 12 & 13 & 10 \end{vmatrix}}{|A|} =$$

$$y = \frac{\begin{vmatrix} 2 & 5 & 3 \\ 9 & 7 & 8 \\ 11 & 12 & 10 \end{vmatrix}}{|A|} =$$

$$z = \frac{\begin{vmatrix} 2 & 4 & 5 \\ 9 & 6 & 7 \\ 11 & 13 & 12 \end{vmatrix}}{|A|} =$$

32.2.2 Gauss-Jordan Row Reduction

- In many ways Gauss-Jordan is a form of substitution based upon rearranging equations into an upper-right triangular form.
- The general method works top to bottom ......

Given,

$$x + y = 5$$

$$2x + 3y = 8$$

The equations can be written in matrix form,

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

eliminate values in the first column by multiplying by a factor and subtracting from the first row,

$$\begin{matrix} \\ -\frac{1}{2} \end{matrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 - \frac{1}{2}2 & 1 - \frac{1}{2}3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 5 - \frac{1}{2}8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Finally, solve for the values starting with the last row and work towards the top row.

$$-0.5y = 1 \qquad y = -2$$

$$1x + 1(-2) = 5 \qquad x = 7$$

32.2.3 Cramer's Rule

Given,

$$x + y = 5$$

$$2x + 3y = 8$$

The equations can be written in matrix form,

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

To find x,

$$x = \frac{\begin{vmatrix} 5 & 1 \\ 8 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{15 - 8}{3 - 2} = 7$$

To find y,

$$y = \frac{\begin{vmatrix} 1 & 5 \\ 2 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{8 - 10}{3 - 2} = -2$$

```
x = det([5 1 ; 8 3]) / det([1 1 ; 2 3])
y = det([1 5 ; 2 8]) / det([1 1 ; 2 3])
```

32.2.4 Triple Product

When we want to do a cross product, followed by a dot product (called the mixed triple product), we can do both steps in one operation by finding the determinant of the following. An example of a problem that would use this shortcut is when a moment is found about one point on a pipe, and then the moment component twisting the pipe is found using the dot product.

$$(d \times F) \bullet u = \begin{vmatrix} u_x & u_y & u_z \\ d_x & d_y & d_z \\ F_x & F_y & F_z \end{vmatrix}$$

32.2.5 Gauss-Siedel

32.3 Problems

1. Perform the vector operations below,

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

Cross Product $A \times B =$

Dot Product $A \bullet B =$

ANS.

$A \times B = (-4, 17, -10)$

$A \bullet B = 13$

2. Perform the following matrix calculations.

a) $$\begin{bmatrix} a & b & c \end{bmatrix}^T \begin{bmatrix} d & e & f \\ g & h & k \\ m & n & p \end{bmatrix}$$

ans.

not solvable

b) $$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right|$$

$ad - bc$

c) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$

$$\frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$$

3. Perform the matrix operations below.

$$\begin{vmatrix} 2 & 0 \\ -6 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 1 \\ -6 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix}$$

ans.

=6

=12

=0

Multiply

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} =$$

Determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$$

Inverse

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{-1} =$$

ANS.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 68 \\ 167 \\ 266 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 36$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} -0.833 & 0.167 & 0.167 \\ 0.167 & -0.333 & 0.167 \\ 0.5 & 0.167 & -0.167 \end{bmatrix}$$

4. Solve the following equations using matrices,

$$\begin{aligned} 5x - 2y + 4z &= -1 \\ 6x + 7y + 5z &= -2 \\ 2x - 3y + 6z &= -3 \end{aligned}$$

ANS.

x= 0.273
y= -0.072
z= -0.627

5. Solve the following system of equations using a) substitution, b) matrices.

$$x + 2y + 3z = 5$$
$$x + 4y + 8z = 0$$
$$4x + 2y + z = 1$$

ans. $x = -7$

$y = 18.75$

$z = -8.5$

6. Find the dot product, and the cross product, of the vectors A and B below.

$$A = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

ans. $A \bullet B = xp + yq + zr$

$$A \times B = \begin{bmatrix} yr - qz \\ zp - xr \\ xq - py \end{bmatrix}$$