

1. ANALYSIS OF DIFFERENTIAL EQUATIONS

Topics:

- First and second-order homogeneous differential equations
- Non-homogeneous differential equations
- First and second-order responses
- Non-linear system elements
- Design case

Objectives:

- To develop explicit equations that describe a system response.
- To recognize first and second-order equation forms.

1.1 Introduction

In the previous chapter we derived differential equations of motion for translating systems. These equations can be used to analyze the behavior of the system and make design decisions. The most basic method is to select a standard input type (a forcing function) and initial conditions, and then solve the differential equation. It is also possible to estimate the system response without solving the differential equation as will be discussed later.

Figure 1.1 shows an abstract description of a system. The basic concept is that the system changes the inputs to outputs. Say, for example, that the system to be analyzed is an elevator. Inputs to the system would be the mass of human riders and desired elevator height. The output response of the system would be the actual height of the elevator. For analysis, the system model could be developed using differential equations for the motor, elastic lift cable, mass of the car, etc. A basic test would involve assuming that the elevator starts at the ground floor and must travel to the top floor. Using assumed initial values and input functions the differential equation could be solved to get an explicit equation for elevator height. This output response can then be used as a guide to modify design choices (parameters). In practice, many of the assumptions and tests are mandated by law or by groups such as Underwriters Laboratories (UL), Canadian Standards Association (CSA) and the European Commission (CE).

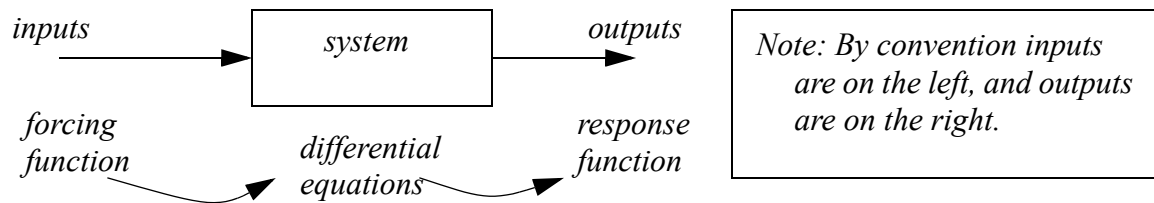


Figure 1.1 A system with and input and output response

There are several standard input types used to test a system. These are listed below in order of relative popularity with brief explanations.

- step - a sudden change of input, such as very rapidly changing a desired speed from 0Hz to 50Hz.
- ramp - a continuously increasing input, such as a motor speed that increases constantly at 10Hz per minute.
- sinusoidal - a cyclic input that varies continuously, such as wave height that is continually oscillating at 1Hz.
- parabolic - an exponentially increasing input, such as a motor speed that is 2Hz at 1 second, 4rad/s at 2 seconds, 8rad/s at 3 seconds, etc.

After the system has been modeled, an input type has been chosen, and the initial conditions have been selected, the system can be analyzed to determine its behavior. The most fundamental technique is to integrate the differential equation(s) for the system.

1.2 Explicit Solutions

Solving a differential equation results in an explicit solution. This equation provides the general response as a function of time, but it can also be used to find frequencies and other characteristics of interest. This section will review techniques used to integrate first and second-order homogenous differential equations. These equations correspond to systems without inputs, also called unforced systems. Non-homogeneous differential equations will also be reviewed.

The basic types of differential equations are shown in Figure 1.2. Each of these equations is linear. On the left hand side is the integration variable 'x'. If the right hand side is zero, then the equation is homogeneous. Each of these equations is linear because each of the terms on the left hand side is simply multiplied by a linear coefficient.

$A\dot{x} + Bx = 0$	<i>first-order homogeneous</i>
$A\dot{x} + Bx = Cf(t)$	<i>first-order non-homogeneous</i>
$A\ddot{x} + B\dot{x} + Cx = 0$	<i>second-order homogeneous</i>
$A\ddot{x} + B\dot{x} + Cx = Df(t)$	<i>second-order non-homogeneous</i>

Figure 1.2 Standard equation forms

A general solution for a first-order homogeneous differential equation is given in Figure 1.3. The solution begins with the solution of the homogeneous equation where a general form is 'guessed'. Substitution leads to finding the value of the coefficient 'Y'. Following this, the initial conditions for the equation are used to find the value of the coefficient 'X'. Notice that the final equation will begin at the initial displacement, but approach zero as time goes to infinity. The e-to-the-x behavior is characteristic for a first-order response.

Given the general form of a first-order homogeneous equation,

$$A\dot{x} + Bx = 0 \quad \text{and} \quad x(0) = x_0$$

Guess a solution form and solve.

$$x = Xe^{-Yt} \quad \dot{x} = -YXe^{-Yt}$$

$$A(-YXe^{-Yt}) + B(Xe^{-Yt}) = 0$$

$$A(-Y) + B = 0$$

$$Y = \frac{B}{A}$$

Therefore the general form is,

$$x_h = Xe^{\frac{-B}{A}t}$$

Next, use the initial conditions to find the remaining unknowns.

$$x_h = Xe^{\frac{-B}{A}t}$$

$$x_0 = Xe^{\frac{-B}{A}0}$$

$$x_0 = X$$

Therefore the final equation is,

$$x(t) = x_0 e^{\frac{-B}{A}t}$$

initial condition

Note: The general form below is useful for finding almost all homogeneous equations

$$x_h(t) = Xe^{-Yt}$$

Figure 1.3 General solution of a first-order homogeneous equation

Solve the following differential equation given the initial condition.

$$\dot{x} + 2x = 0$$

$$x(0) = 3$$

ans. $x(t) = 3e^{-2t}$

Figure 1.4 Drill Problem: First order homogeneous differential equation

The general solution to a second-order homogeneous equation is shown in Figure 1.5. The solution begins with a guess of the homogeneous solution, and the solution of a quadratic equation. There are three possible cases that result from the solution of the quadratic equation: different but real roots; two identical real roots; or two complex roots. The three cases result in three different forms of solutions, as shown. The complex result is the most notable because it results in sinusoidal oscillations. It is not shown, but after the homogeneous solution has been found, the initial conditions need to be used to find the remaining coefficient values.

Given,

$$A\ddot{x} + B\dot{x} + Cx = 0 \quad x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0$$

Guess a general equation form and substitute it into the differential equation,

$$x_h = X e^{Yt} \quad \dot{x}_h = Y X e^{Yt} \quad \ddot{x}_h = Y^2 X e^{Yt}$$

$$A(Y^2 X e^{Yt}) + B(Y X e^{Yt}) + C(X e^{Yt}) = 0$$

$$A(Y^2) + B(Y) + C = 0$$

$$Y = \frac{-B \pm \sqrt{(B)^2 - 4(AC)}}{2A} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Note: There are three possible outcomes of finding the roots of the equations: two different real roots, two identical real roots, or two complex roots. Therefore there are three fundamentally different results.

If the values for Y are both real, but different, the general form is,

$$Y = R_1, R_2 \quad x_h = X_1 e^{R_1 t} + X_2 e^{R_2 t}$$

Note: The initial conditions are then used to find the values for X_1 and X_2 .

If the values for Y are both real, and identical, the general form is,

$$Y = R_1, R_1 \quad x_h = X_1 e^{R_1 t} + X_2 t e^{R_1 t}$$

The initial conditions are then used to find the values for X_1 and X_2 .

If the values for Y are complex, the general form is,

$$Y = \sigma \pm \omega j \quad x_h = X_3 e^{\sigma t} \cos(\omega t + X_4)$$

The initial conditions are then used to find the values of X_3 and X_4 .

Figure 1.5 Solution of a second-order homogeneous equation

As mentioned above, a complex solution when solving the homogeneous equation results in a sinusoidal oscillation, as proven in Figure 1.6. The most notable part of the solution is that there is both a frequency of oscillation and a phase shift. This form is very useful for analyzing the frequency response of a system, as will be seen in a later chapter.

Consider the situation where the results of a homogeneous solution are the complex conjugate pair.

$$Y = R \pm Cj$$

This gives the general result, as shown below:

$$x = X_1 e^{(R + Cj)t} + X_2 e^{(R - Cj)t}$$

$$x = X_1 e^{Rt} e^{Cjt} + X_2 e^{Rt} e^{-Cjt}$$

$$x = e^{Rt} (X_1 e^{Cjt} + X_2 e^{-Cjt})$$

$$x = e^{Rt} (X_1 (\cos(Ct) + j \sin(Ct)) + X_2 (\cos(-Ct) + j \sin(-Ct)))$$

$$x = e^{Rt} (X_1 (\cos(Ct) + j \sin(Ct)) + X_2 (\cos(Ct) - j \sin(Ct)))$$

$$x = e^{Rt} ((X_1 + X_2) \cos(Ct) + j(X_1 - X_2) \sin(Ct))$$

$$x = e^{Rt} ((X_1 + X_2) \cos(Ct) + j(X_1 - X_2) \sin(Ct))$$

$$x = e^{Rt} \frac{\sqrt{(X_1 + X_2)^2 + j^2 (X_1 - X_2)^2}}{\sqrt{(X_1 + X_2)^2 + j^2 (X_1 - X_2)^2}} ((X_1 + X_2) \cos(Ct) + j(X_1 - X_2) \sin(Ct))$$

$$x = e^{Rt} \frac{\sqrt{X_1^2 + 2X_1X_2 + X_2^2 - (X_1^2 - 2X_1X_2 + X_2^2)}}{\sqrt{X_1^2 + 2X_1X_2 + X_2^2 - (X_1^2 - 2X_1X_2 + X_2^2)}} ((X_1 + X_2) \cos(Ct) + j(X_1 - X_2) \sin(Ct))$$

$$x = e^{Rt} \frac{\sqrt{4X_1X_2}}{\sqrt{4X_1X_2}} ((X_1 + X_2) \cos(Ct) + j(X_1 - X_2) \sin(Ct))$$

$$x = e^{Rt} \sqrt{4X_1X_2} \left(\frac{(X_1 + X_2)}{\sqrt{4X_1X_2}} \cos(Ct) + j \frac{(X_1 - X_2)}{\sqrt{4X_1X_2}} \sin(Ct) \right)$$

$$x = e^{Rt} \sqrt{4X_1X_2} \cos \left(Ct + \operatorname{atan} \left(\frac{(X_1 - X_2)}{(X_1 + X_2)} \right) \right)$$

$$x = e^{Rt} X_3 \cos(Ct + X_4)$$

frequency

phase shift

$$\text{where, } X_3 = \sqrt{4X_1X_2}$$

$$X_4 = \operatorname{atan} \left(\frac{(X_1 - X_2)}{(X_1 + X_2)} \right)$$

Figure 1.6 Phase shift solution for a second-order homogeneous differential equation

Note: Occasionally a problem solution might consist of both a sine and cosine term with the same frequency. These should normally be combined to a single term with a phase shift as shown below.

Recall the double angle formula,

$$\sin(\omega t + \theta) = \sin \omega t \cos \theta + \sin \theta \cos \omega t$$

This can be written in a more common form,

$$A(\sin \omega t \cos \theta + \sin \theta \cos \omega t) = A \sin(\omega t + \theta)$$

$$A \cos \theta \sin \omega t + A \sin \theta \cos \omega t = A \sin(\omega t + \theta)$$

$$B \sin \omega t + C \cos \omega t = A \sin(\omega t + \theta) \quad \text{where,}$$

$$B = A \cos \theta$$

$$C = A \sin \theta$$

$$A = \frac{B}{\cos \theta} = \frac{C}{\sin \theta}$$

$$\frac{\sin \theta}{\cos \theta} = \frac{C}{B}$$

$$\theta = \operatorname{atan}\left(\frac{C}{B}\right)$$

$$A = \sqrt{B^2 + C^2}$$

Consider the example,

$$3 \sin 5t + 4 \cos 5t = \sqrt{3^2 + 4^2} \sin\left(5t + \operatorname{atan}\left(\frac{4}{3}\right)\right) = 5 \sin(5t + 0.927)$$

Figure 1.7 Phase shift solution form

Solve the following differential equation given the initial condition.

$$\ddot{x} + 2\dot{x} + x = 0$$

$$x(0) = 1$$

$$\dot{x}(0) = 2$$

ans. $x(t) = e^{-t} + 3te^{-t}$

Figure 1.8 Drill Problem: Second order homogeneous differential equation

The methods for solving non-homogeneous differential equations builds upon the methods used for the solution of homogeneous equations. This process adds a step to find the particular solution of the equation. An example of the solution of a first-order non-homogeneous equation is shown in Figure 1.9. To find the homogeneous solution the non-homogeneous part of the equation is set to zero. To find the particular solution the final

form must be guessed. This is then substituted into the equation, and the values of the coefficients are found. Finally the homogeneous and particular solutions are added to get the final equation. The overall response of the system can be obtained by adding the homogeneous and particular parts. This is acceptable because the equations are linear, and the principle of superposition applies. The homogeneous equation deals with the response to initial conditions, and the particular solution deals with the response to forced inputs.

Generally,

$$A\dot{x} + Bx = Cf(t) \quad x(0) = x_0$$

First, find the homogeneous solution as before, in Figure 1.3.

$$x_h = x_0 e^{-\frac{B}{A}t}$$

Next, guess the particular solution by looking at the form of 'f(t)'. This step is highly subjective, and if an incorrect guess is made, it will be unsolvable. When this happens, just make another guess and repeat the process. An example is given below. In the case below the guess should be similar to the exponential forcing function.

For example, if we are given

$$6\dot{x} + 2x = 5e^{4t}$$

A reasonable guess for the particular solution is,

$$x_p = C_1 e^{4t} \quad \dot{x}_p = 4C_1 e^{4t}$$

Substitute these into the differential equation and solve for A.

$$6(4C_1 e^{4t}) + 2(C_1 e^{4t}) = 5e^{4t}$$

$$24C_1 + 2C_1 = 5 \quad \therefore C_1 = \frac{5}{26}$$

Combine the particular and homogeneous solutions.

$$x = x_p + x_h = \frac{5}{26}e^{4t} + x_0 e^{-\frac{6}{2}t}$$

Figure 1.9 Solution of a first-order non-homogeneous equation

The method for finding a particular solution for a second-order non-homogeneous differential equation is shown in Figure 1.10. In this example the forcing function is sinusoidal, so the particular result should also be sinusoidal. The final result is converted into a phase shift form.

Generally,

$$A\ddot{x} + B\dot{x} + Cx = Df(t) \quad x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0$$

1. Find the homogeneous solution as before.

$$\begin{aligned} x_h &= X_3 e^{\sigma t} \cos(\omega t + X_4) \\ \text{or } x_h &= X_1 e^{\sigma_1 t} + X_2 t e^{\sigma_2 t} \\ \text{or } x_h &= X_1 e^{\sigma_1 t} + X_2 e^{\sigma_2 t} \end{aligned}$$

2. Guess the particular solution by looking at the form of 'f(t)'. This step is highly subjective, and if an incorrect guess is made it will be unsolvable. When this happens, just make another guess and repeat the process. For the purpose of illustration an example is given below. In the case below it should be similar to the sine function.

For example, if we are given

$$2\ddot{x} + 6\dot{x} + 2x = 2\sin(3t + 4)$$

A reasonable guess is,

$$\begin{aligned} x_p &= A\sin(3t) + B\cos(3t) \\ \dot{x}_p &= 3A\cos(3t) - 3B\sin(3t) \\ \ddot{x}_p &= -9A\sin(3t) - 9B\cos(3t) \end{aligned}$$

Substitute these into the differential equation and solve for A and B.

$$\begin{aligned} 2(-9A\sin(3t) - 9B\cos(3t)) + 6(3A\cos(3t) - 3B\sin(3t)) + \\ 2(A\sin(3t) + B\cos(3t)) = 2\sin(3t + 4) \end{aligned}$$

$$(-18A - 18B + 2A)\sin(3t) + (-18B + 18A + 2B)\cos(3t) = 2\sin(3t + 4)$$

$$(-16A - 18B)\sin(3t) + (18A - 16B)\cos(3t) = 2(\sin 3t \cos 4 + \cos 3t \sin 4)$$

$$(-16A - 18B)\sin(3t) + (18A - 16B)\cos(3t) = (2\cos 4)\sin(3t) + (2\sin 4)\cos(3t)$$

$$-16A - 18B = 2\cos 4 \quad 18A - 16B = 2\sin 4$$

$$\begin{bmatrix} -16 & -18 \\ 18 & -16 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2\cos 4 \\ 2\sin 4 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -16 & -18 \\ 18 & -16 \end{bmatrix}^{-1} \begin{bmatrix} -1.307 \\ -1.514 \end{bmatrix} = \begin{bmatrix} -0.0109 \\ 0.0823 \end{bmatrix}$$

Next, rearrange the equation to phase shift form.

$$x_p = -0.0109\sin(3t) + 0.0823\cos(3t)$$

$$x_p = \sqrt{-0.0109^2 + 0.0823^2} \sin\left(3t + \operatorname{atan}\left(\frac{0.0823}{-0.0109}\right) + \frac{\pi}{2}\right)$$

3. Use the initial conditions to determine the coefficients in the homogeneous solution.

Figure 1.10 Solution of a second-order non-homogeneous equation

When guessing particular solutions, the forms in Figure 1.11 can be helpful.

<i>Forcing Function</i>	<i>Guess</i>
A	C
$Ax + B$	$Cx + D$
e^{Ax}	Ce^{Ax} or Cxe^{Ax}
$B\sin(Ax)$ or $B\cos(Ax)$	$C\sin(Ax) + D\cos(Ax)$ or $Cx\sin(Ax) + xD\cos(Ax)$

Figure 1.11 General forms for particular solutions

Solve the following differential equation given the initial condition.

$$\ddot{x} + 2\dot{x} + x = 1$$

$$x(0) = 0$$

$$\dot{x}(0) = 0$$

$$\text{ans. } x(t) = -e^{-t} - te^{-t} + 1$$

Figure 1.12 Drill Problem: Second order non-homogeneous differential equation

An example of a second-order system is shown in Figure 1.13. As expected, it begins with a FBD and summation of forces. This is followed with the general solution of

the homogeneous equation. Real roots are assumed thus allowing the problem solution to continue in Figure 1.14.

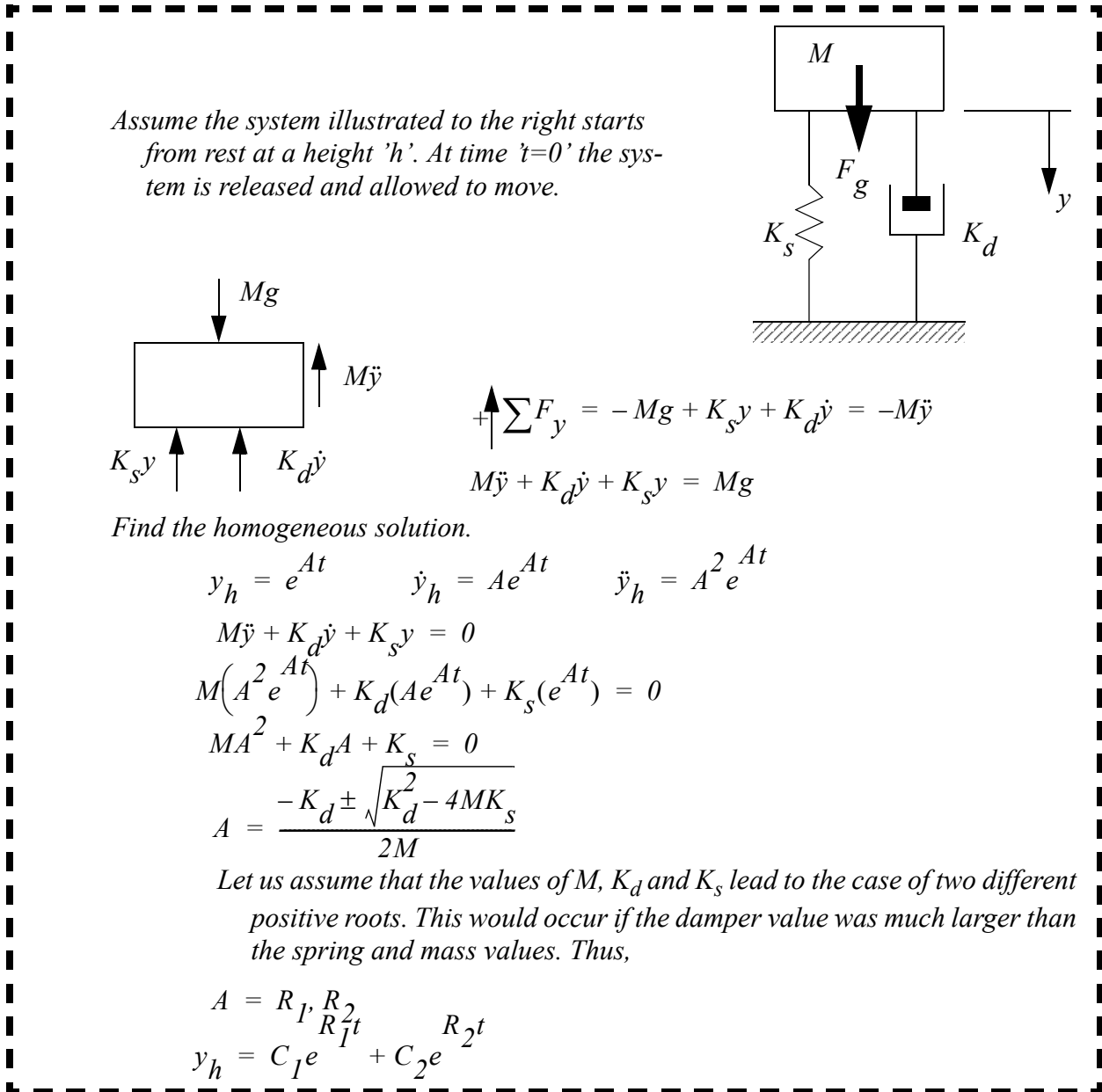


Figure 1.13 Second-order system example

The solution continues by assuming a particular solution and calculating values for the coefficients using the initial conditions in Figure 1.14. The final result is a second-order system that is overdamped, with no oscillation.

Next, find the particular solution.

$$y_p = C \quad \dot{y}_h = 0 \quad \ddot{y}_h = 0$$

$$M(0) + K_d(0) + K_s(C) = Mg$$

$$C = \frac{Mg}{K_s}$$

Now, add the homogeneous and particular solutions and solve for the unknowns using the initial conditions.

$$y(t) = y_p + y_h = \frac{Mg}{K_s} + C_1 e^{R_1 t} + C_2 e^{R_2 t}$$

$$y(0) = h \quad y'(0) = 0$$

$$h = \frac{Mg}{K_s} + C_1 e^0 + C_2 e^0$$

$$C_1 + C_2 = h - \frac{Mg}{K_s}$$

$$y'(t) = R_1 C_1 e^{R_1 t} + R_2 C_2 e^{R_2 t}$$

$$0 = R_1 C_1 e^0 + R_2 C_2 e^0$$

$$0 = R_1 C_1 + R_2 C_2 \quad C_1 = \frac{-R_2}{R_1} C_2$$

$$-\frac{R_2}{R_1} C_2 + C_2 = h - \frac{Mg}{K_s}$$

$$C_2 = \left(\frac{K_s h - Mg}{K_s} \right) \left(\frac{-R_1}{R_1 - R_2} \right) \quad C_1 = \frac{-R_2}{R_1} \left(\frac{K_s h - Mg}{K_s} \right) \left(\frac{-R_1}{R_1 - R_2} \right)$$

Now, combine the solutions and solve for the unknowns using the initial conditions.

$$y(t) = \frac{Mg}{K_s} + \left(\frac{K_s h - Mg}{K_s} \right) \left(\frac{-R_1}{R_1 - R_2} \right) e^{R_1 t} + \frac{-R_2}{R_1} \left(\frac{K_s h - Mg}{K_s} \right) \left(\frac{-R_1}{R_1 - R_2} \right) e^{R_2 t}$$

$$y(t) = \frac{Mg}{K_s} + \left(\frac{K_s h - Mg}{K_s} \right) \left(\frac{-R_1}{R_1 - R_2} \right) e^{R_1 t} + \left(\frac{K_s h - Mg}{K_s} \right) \left(\frac{R_2}{R_1 - R_2} \right) e^{R_2 t}$$

Figure 1.14 Second-order system example (continued)

Given,

$$A \sin(\omega t + \theta) \quad (\text{the desired final form})$$

$$A(\cos \omega t \sin \theta + \sin \omega t \cos \theta)$$

$$(A \sin \theta) \cos \omega t + (A \cos \theta) \sin \omega t$$

$$B \cos \omega t + C \sin \omega t \quad (\text{the form we start with})$$

where,

$$B = A \sin \theta$$

$$C = A \cos \theta$$

To find theta,

$$\frac{B}{C} = \frac{A \sin \theta}{A \cos \theta} = \tan \theta$$

$$\theta = \text{atan}\left(\frac{B}{C}\right)$$

To find A, (method #1)

$$A = \frac{B}{\sin \theta} = \frac{C}{\cos \theta}$$

To find A, (method #2)

$$A = \sqrt{B^2 + C^2}$$

For example,

$$3 \cos 5t + 4 \sin 5t$$

$$\sqrt{3^2 + 4^2} \sin\left(5t + \text{atan}\frac{3}{4}\right)$$

$$5 \sin(5t + 0.6435)$$

Figure 1.15 Proof for conversion to phase form

1.3 Responses

Solving differential equations tends to yield one of two basic equation forms. The e-to-the-negative-t forms are the first-order responses and slowly decay over time. They never naturally oscillate, and only oscillate if forced to do so. The second-order forms may

include natural oscillation.

1.3.1 First-order

A first-order system is described with a first-order differential equation. The response function for these systems is natural decay or growth as shown in Figure 1.16. The time constant for the system can be found directly from the differential equation. It is a measure of how quickly the system responds to a change. When an input to a system has changed, the system output will be approximately 63% of the way to its final value when the elapsed time equals the time constant. The initial and final values of the function can be determined algebraically to find the first-order response with little effort.

If we have experimental results for a system, we can calculate the time constant, initial and final values. The time constant can be found two ways, one by extending the slope of the first part of the curve until it intersects the final value line. That time at the intersection is the time constant. The other method is to look for the time when the output value has shifted 63.2% of the way from the initial to final values for the system. Assuming the change started at $t=0$, the time at this point corresponds to the time constant.

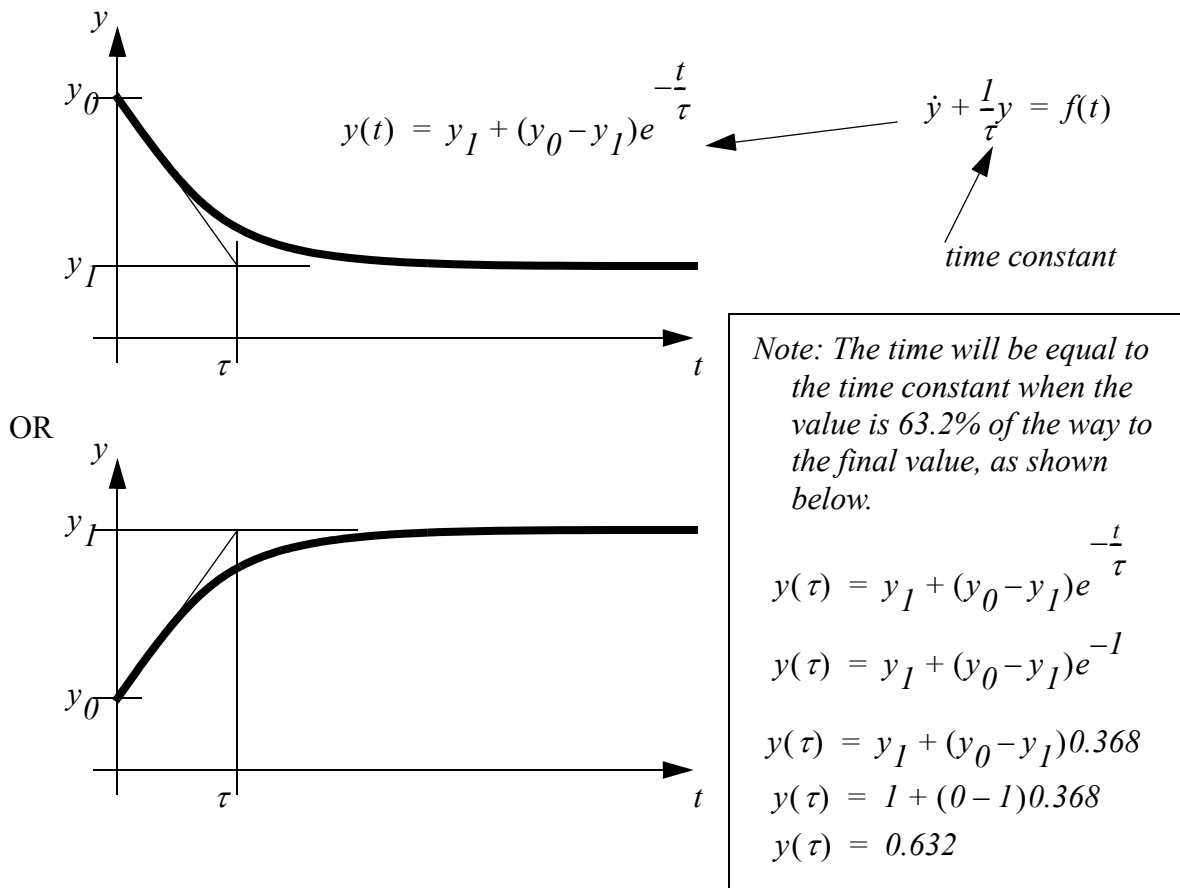
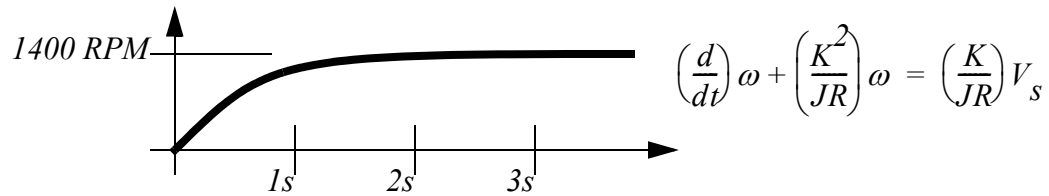


Figure 1.16 Typical first-order responses

The example in Figure 1.17 calculates the coefficients for a first-order differential equation given a graphical output response to an input. The differential equation is for a permanent magnet DC motor, and will be examined in a later chapter. If we consider the steady state when the speed is steady at 1400RPM, the first derivative will be zero. This simplifies the equation and allows us to calculate a value for the parameter K in the differential equation. The time constant can be found by drawing a line asymptotic to the start of the motor curve, and finding the point where it intercepts the steady-state value. This gives an approximate time constant of 0.8 s. This can then be used to calculate the remaining coefficient. Some additional numerical calculation leads to the final differential equation as shown.

For the motor, use the differential equation and the speed curve when $V_s=10V$ is applied:

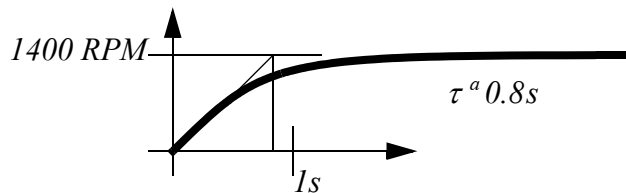


For steady-state

$$\left(\frac{d}{dt}\right)\omega = 0 \quad \omega = 1400\text{RPM} = 146.6\text{rads}^{-1}$$

$$0 + \left(\frac{K^2}{JR}\right)146.6 = \left(\frac{K}{JR}\right)10$$

$$K = 0.0682$$



$$\left(\frac{K^2}{JR}\right) = \frac{1}{0.8s}$$

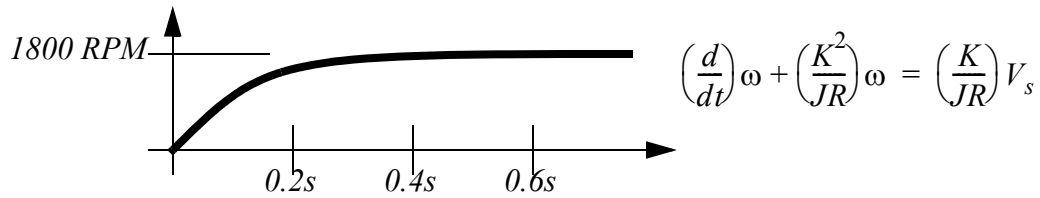
$$0.0682\left(\frac{K}{JR}\right) = \frac{1}{0.8s}$$

$$\frac{K}{JR} = 18.328$$

$$\dot{\omega} + \frac{1}{0.8}\omega = 18.328V_s$$

Figure 1.17 Finding an equation using experimental data

Find the differential equation when a step input of $V_s=12V$ is applied:

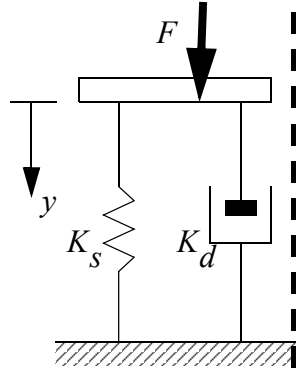
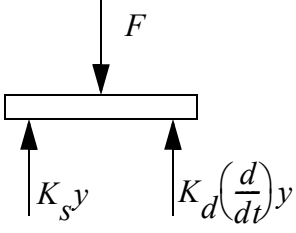


ans. $\left(\frac{d}{dt}\right)\omega + \frac{1}{0.15}\omega = \frac{1800}{12(0.15)}V_s$

Figure 1.18 Drill problem: Find the constants for the equation

A simple mechanical example is given in Figure 1.19. The modeling starts with a FBD and a sum of forces. After this, the homogenous solution is found by setting the non-homogeneous part to zero and solving. Next, the particular solution is found, and the two solutions are combined. The initial conditions are used to find the remaining unknown coefficients.

Find the response to the applied force if the force is applied at $t=0s$. Assume the system is initially deflected a height of h .

$$+\uparrow \sum F_y = -F + K_s y + K_d \left(\frac{d}{dt} \right) y = 0$$

$$K_d \dot{y} + K_s y = F$$

Find the homogeneous solution.

$$y_h = Ae^{Bt} \quad \dot{y}_h = AB e^{Bt}$$

$$K_d (AB e^{Bt}) + K_s (A e^{Bt}) = 0$$

$$K_d B + K_s = 0$$

$$B = \frac{-K_s}{K_d}$$

Next, find the particular solution.

$$y_p = C \quad \dot{y}_p = 0$$

$$K_d(0) + K_s(C) = F \quad \therefore C = \frac{F}{K_s}$$

Combine the solutions, and find the remaining unknown.

$$y(t) = y_p + y_h = Ae^{\frac{-K_s}{K_d}t} + \frac{F}{K_s}$$

$$y(0) = h$$

$$h = Ae^0 + \frac{F}{K_s} \quad \therefore A = h - \frac{F}{K_s}$$

The final solution is,

$$y(t) = \left(h - \frac{F}{K_s} \right) e^{\frac{-K_s}{K_d}t} + \frac{F}{K_s}$$

Figure 1.19 First-order system analysis example

Use the general form given below to solve the problem in Figure 1.19 without solving the differential equation. Assume the system starts at $y=-20$.

$$\dot{y} + 10y = 5 \longrightarrow y(t) = y_1 + (y_0 - y_1)e^{-\frac{t}{\tau}}$$

$$\text{ans. } y(t) = 0.5 - 19.5e^{-10t}$$

Figure 1.20 Drill problem: Developing the final equation using the first-order model form

A first-order system tends to be passive, meaning it doesn't deliver energy or power. A first-order system will not oscillate unless the input forcing function is also oscillating. The output response lags the input and the delay is determined by the system's time constant.

1.3.2 Second-order

A second-order system response typically contains two first-order responses, or a first-order response and a sinusoidal component. A typical sinusoidal second-order response is shown in Figure 1.21. Notice that the coefficients of the differential equation include a damping coefficient and a natural frequency. These can be used to develop the final response, given the initial conditions and forcing function. Notice that the damped frequency of oscillation is the actual frequency of oscillation. The damped frequency will be lower than the natural frequency when the damping coefficient is between 0 and 1. If the damping coefficient is greater than one the damped frequency becomes negative, and the system will not oscillate because it is overdamped.

A second-order system, and a typical response to a stepped input.

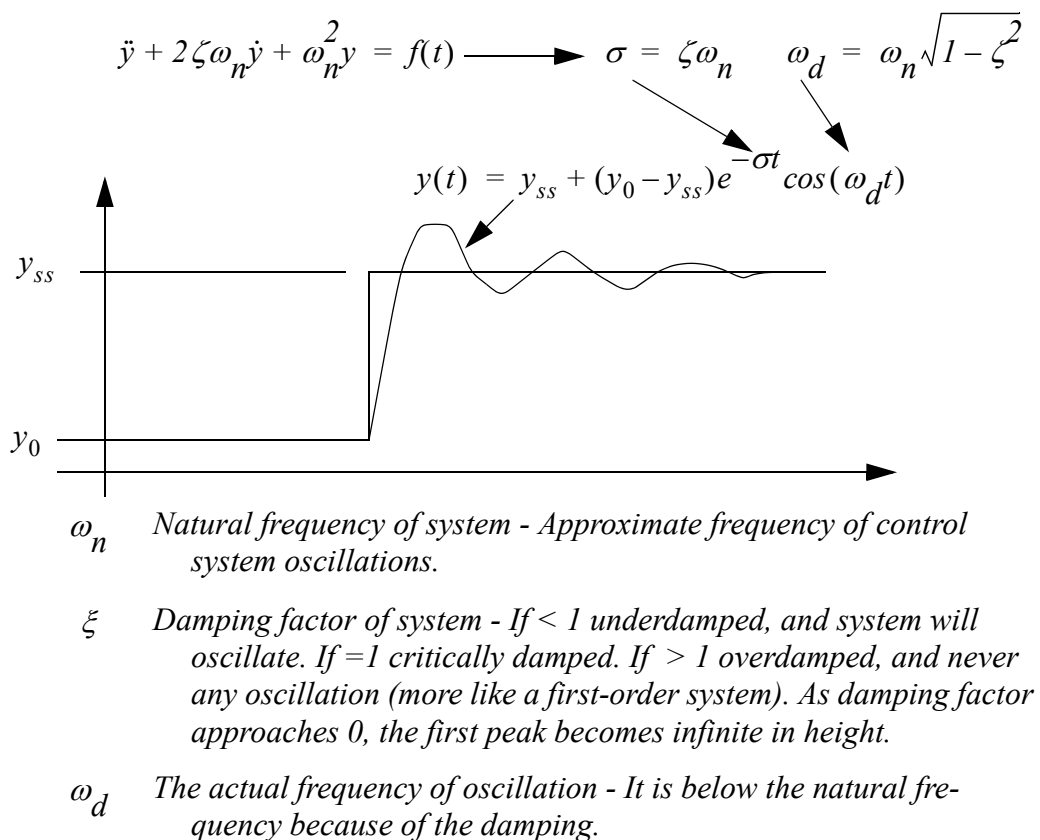


Figure 1.21 The general form for a second-order system

When only the damping coefficient is increased, the frequency of oscillation, and overall response time will slow, as seen in Figure 1.22. When the damping coefficient is 0 the system will oscillate indefinitely. Critical damping occurs when the damping coeffi-

cient is 1. At this point both roots of the differential equation are equal. The system will not oscillate if the damping coefficient is greater than or equal to 1.

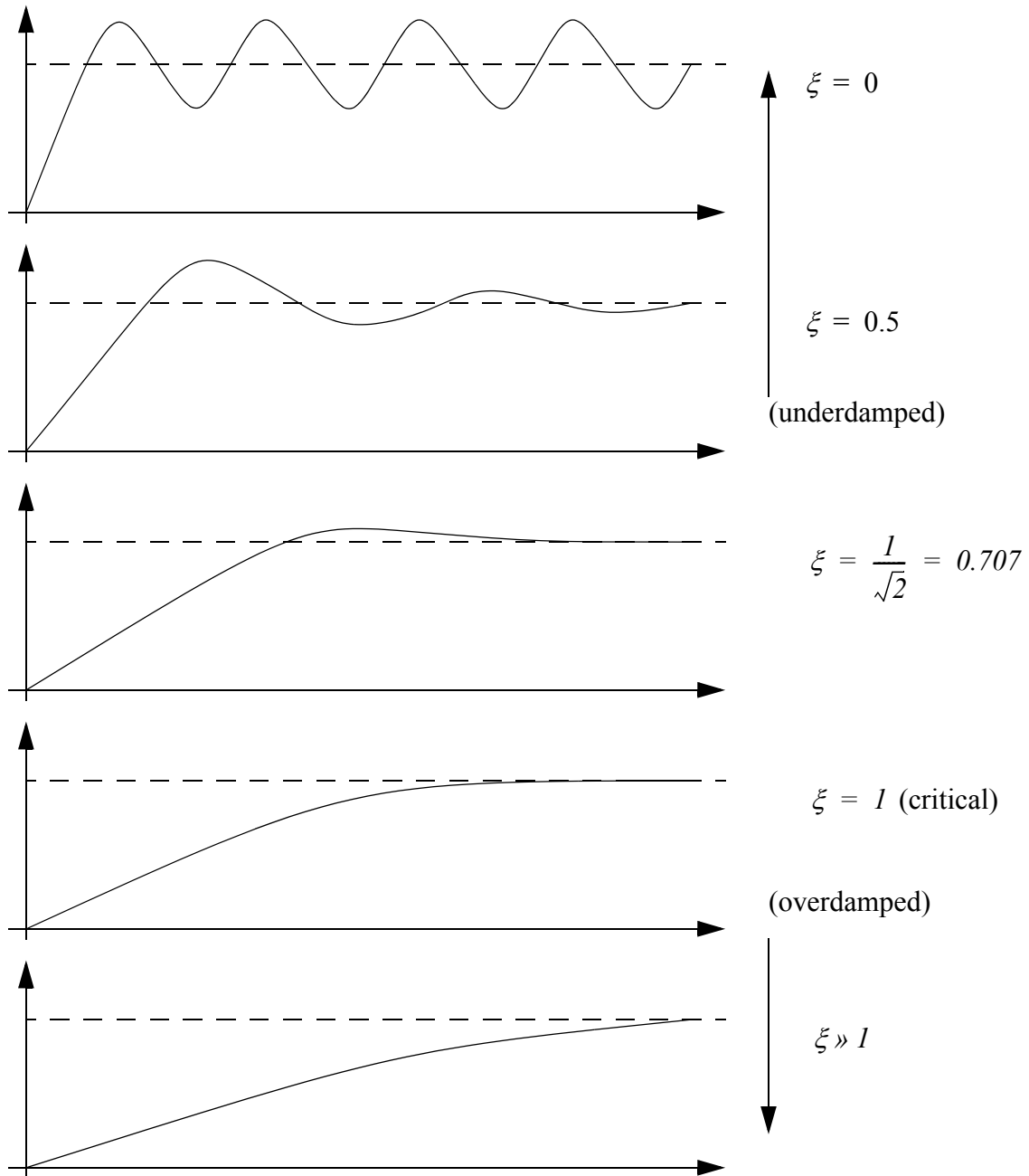


Figure 1.22 The effect of the damping coefficient

When observing second-order systems it is more common to use more direct measurements of the response. Some of these measures are shown in Figure 1.23. The rise

time is the time it takes to go from 10% to 90% of the total displacement, and is comparable to a first order time constant. The settling time indicates how long it takes for the system to pass within a tolerance band around the final value. The permissible zone shown is 2%, but if it were larger the system would have a shorter settling time. The period of oscillation can be measured directly as the time between peaks of the oscillation, the inverse is the damping frequency. (Note: don't forget to convert to radians.) The damped frequency can also be found using the time to the first peak, as half the period. The overshoot is the height of the first peak. Using the time to the first peak, and the overshoot the damping coefficient can be found.

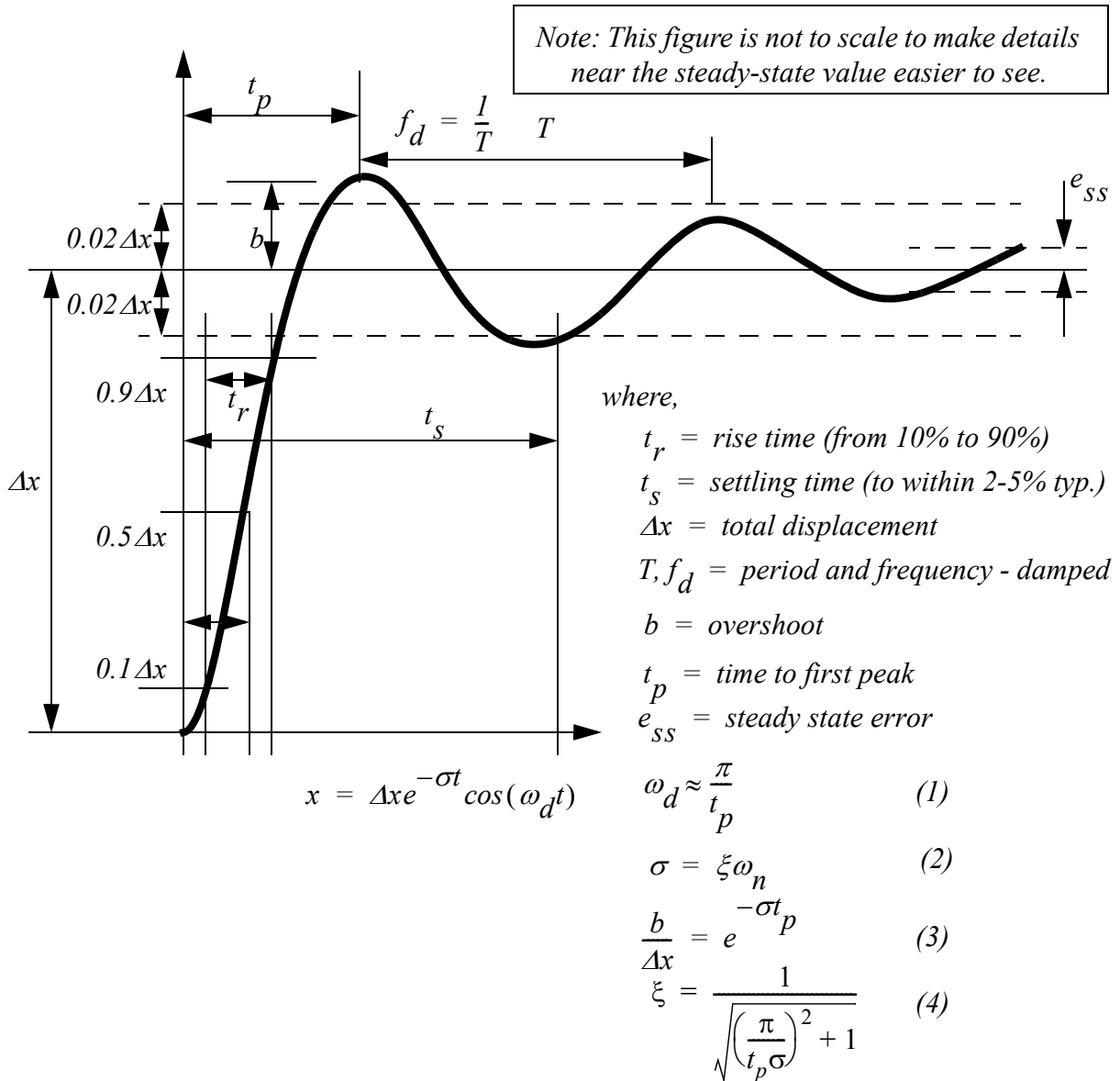


Figure 1.23 Characterizing a second-order response (not to scale)

Note: We can calculate these relationships using the complex homogenous form, and the generic second order equation form.

$$A^2 + 2\xi\omega_n A + \omega_n^2 = 0$$

$$A = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = \sigma \pm j\omega_d$$

$$\frac{-2\xi\omega_n}{2} = \sigma = -\xi\omega_n \quad \omega_n = \frac{\sigma}{-\xi} \quad (1)$$

$$\frac{\sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = j\omega_d$$

$$4\xi^2\omega_n^2 - 4\omega_n^2 = 4(-1)\omega_d^2$$

$$\omega_n^2 - \xi^2\omega_n^2 = \omega_d^2 \quad \omega_n\sqrt{1-\xi^2} = \omega_d \quad (2)$$

$$\frac{\sigma^2}{\xi^2} - \xi^2\frac{\sigma^2}{\xi^2} = \omega_d^2$$

$$\frac{1}{\xi^2} = \frac{\omega_d^2}{\sigma^2} + 1 \quad \xi = \frac{1}{\sqrt{\frac{\omega_d^2}{\sigma^2} + 1}} \quad (3)$$

The time to the first peak can be used to find the approximate decay constant

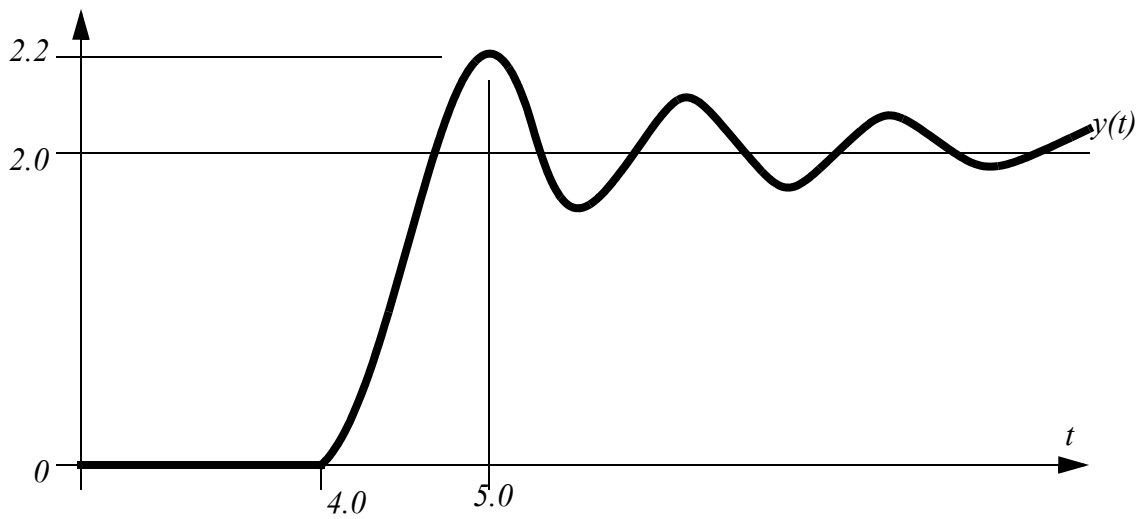
$$x(t) = C_1 e^{-\sigma t} \cos(\omega_d t + C_2)$$

$$\omega_d = \frac{\pi}{t_p} \quad (4)$$

$$b \approx \Delta x e^{-\sigma t_p} (1)$$

$$\sigma = -\frac{\ln\left(\frac{b}{\Delta x}\right)}{t_p} \quad (5)$$

Figure 1.24 Second order relationships between damped and natural frequency



Write a function of time for the graph. (Note: measure, using a ruler, to get values.) Find the natural frequency and damping coefficient to develop the differential equation. Using the dashed lines determine the settling time.

ans.

$$t < 4 \quad y(t) = 0$$

$$t \geq 4 \quad y(t) =$$

Figure 1.25 Drill problem: Find the equation given the response curve

1.3.3 Other Responses

First-order systems have e-to-the-t type responses. Second-order systems add another e-to-the-t response or a sinusoidal excitation. As we move to higher order linear systems we typically add more e-to-the-t terms, and/or more sinusoidal terms. A possible higher order system response is seen in Figure 1.26. The underlying function is a first-order response that drops at the beginning, but levels out. There are two sinusoidal functions superimposed, one with about one period showing, the other with a much higher frequency.

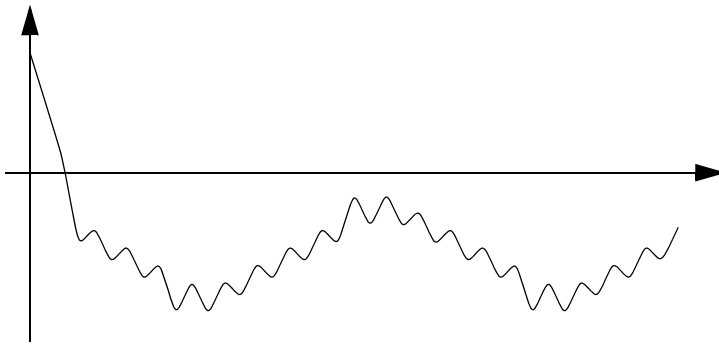


Figure 1.26 An example of a higher order system response

The basic techniques used for solving first and second-order differential equations can be applied to higher order differential equations, although the solutions will start to become complicated for systems with much higher orders. The example in Figure 1.27 shows a fourth order differential equation. In this case the resulting homogeneous solution yields four roots. The result in this case are two real roots, and a complex pair. The two real roots result in e-to-the-t terms, while the complex pair results in a damped sinusoid. The particular solution is relatively simple to find in this example because the non-homogeneous term is a constant.

Given the homogeneous differential equation

$$\left(\frac{d}{dt}\right)^4 x + 13\left(\frac{d}{dt}\right)^3 x + 34\left(\frac{d}{dt}\right)^2 x + 42\left(\frac{d}{dt}\right)x + 20x = 5$$

Guess a solution for the homogeneous equation,

$$x_h = e^{At}$$

$$\frac{d}{dt}x_h = Ae^{At} \quad \left(\frac{d}{dt}\right)^2 x_h = A^2 e^{At} \quad \left(\frac{d}{dt}\right)^3 x_h = A^3 e^{At} \quad \left(\frac{d}{dt}\right)^4 x_h = A^4 e^{At}$$

Substitute the values into the differential equation and find a value for the unknown.

$$A^4 e^{At} + 13A^3 e^{At} + 34A^2 e^{At} + 42Ae^{At} + 20e^{At} = 0$$

$$A^4 + 13A^3 + 34A^2 + 42A + 20 = 0$$

$$A = -1, -10, -1-j, -1+j$$

$$x_h = C_1 e^{-t} + C_2 e^{-10t} + C_3 e^{-t} \cos(t + C_4)$$

Guess a particular solution, and solve for the coefficient.

$$x_p = A \quad \frac{d}{dt}x_p = 0 \quad \left(\frac{d}{dt}\right)^2 x_p = 0 \quad \left(\frac{d}{dt}\right)^3 x_p = 0 \quad \left(\frac{d}{dt}\right)^4 x_p = 0$$

$$0 + 13(0) + 34(0) + 42(0) + 20A = 5 \quad A = 0.25$$

Figure 1.27 Solution of a higher order differential equation

The example is continued in Figure 1.28 and Figure 1.29 where the initial conditions are used to find values for the coefficients in the homogeneous solution.

Solve for the unknowns, assuming the system starts at rest and undeflected.

$$x(t) = C_1 e^{-t} + C_2 e^{-10t} + C_3 e^{-t} \cos(t + C_4) + 0.25$$

$$0 = C_1 + C_2 + C_3 \cos(C_4) + 0.25 \quad (1)$$

$$C_3 \cos(C_4) = -C_1 - C_2 - 0.25 \quad (2)$$

$$\frac{d}{dt} x_h(t) = -C_1 e^{-t} - 10C_2 e^{-10t} - C_3 e^{-t} \cos(t + C_4) - C_3 e^{-t} \sin(t + C_4)$$

$$0 = -C_1 - 10C_2 - C_3 \cos(C_4) - C_3 \sin(C_4) \quad (3)$$

Equations (1) and (3) can be added to get the simplified equation below.

$$0 = -9C_2 - C_3 \sin(C_4) + 0.25$$

$$C_3 \sin(C_4) = -9C_2 + 0.25 \quad (4)$$

$$\left(\frac{d}{dt}\right)^2 x_h(t) = C_1 e^{-t} + 100C_2 e^{-10t} + C_3 e^{-t} \cos(t + C_4) + C_3 e^{-t} \sin(t + C_4) + C_3 e^{-t} \sin(t + C_4) - C_3 e^{-t} \cos(t + C_4)$$

$$0 = C_1 + 100C_2 + C_3 \cos(C_4) + C_3 \sin(C_4) + C_3 \sin(C_4) - C_3 \cos(C_4)$$

$$0 = C_1 + 100C_2 + 2C_3 \sin(C_4) \quad (5)$$

Equations (4) and (5) can be combined.

$$0 = C_1 + 100C_2 + 2(-9C_2 + 0.25)$$

$$0 = -17C_1 + 100C_2 + 0.5 \quad (6)$$

$$\left(\frac{d}{dt}\right)^3 x_h(t) = -C_1 e^{-t} + (-1000)C_2 e^{-10t} - 2C_3 e^{-t} \sin(t + C_4) + 2C_3 e^{-t} \cos(t + C_4)$$

$$0 = -C_1 + (-1000)C_2 - 2C_3 \sin(C_4) + 2C_3 \cos(C_4) \quad (7)$$

Figure 1.28 Solution of a higher order differential equation

Equations (2 and (4) are substituted into equation (7).

$$\begin{aligned}
 0 &= -C_1 + (-1000)C_2 - 2(-9C_2 + 0.25) + 2(-C_1 - C_2 - 0.25) \\
 0 &= -3C_1 + (-984)C_2 - 1 \\
 C_1 &= \left(-\frac{984}{3}\right)C_2 - \frac{1}{3} \tag{8}
 \end{aligned}$$

Equations (6) and (8) can be combined.

$$\begin{aligned}
 0 &= -17\left(\left(-\frac{984}{3}\right)C_2 - \frac{1}{3}\right) + 100C_2 + 0.5 \\
 0 &= 5676C_2 + 6.1666667 & C_2 &= -0.00109 \\
 C_1 &= \left(-\frac{984}{3}\right)(-0.00109) - \frac{1}{3} & C_1 &= 0.0242
 \end{aligned}$$

Equations (2) and (4) can be combined.

$$\begin{aligned}
 \frac{C_3 \sin(C_4)}{C_3 \cos(C_4)} &= \frac{-9C_2 + 0.25}{-C_1 - C_2 - 0.25} \\
 \tan(C_4) &= \frac{-9(-0.00109) + 0.25}{-(0.0242) - (-0.00109) - 0.25} & C_4 &= -0.760
 \end{aligned}$$

Equation (4) can be used to find the remaining unknown.

$$C_3 \sin(-0.760) = -9(-0.00109) + 0.25 \quad C_3 = -0.377$$

The final response function is,

$$x(t) = 0.0242e^{-t} + (-0.00109)e^{-10t} + (-0.377)e^{-t} \cos(t - 0.760) + 0.25$$

Figure 1.29 Solution of a higher order differential equation (cont'd)

In some cases we will have systems with multiple differential equations, or non-linear terms. In these cases explicit analysis of the equations may not be feasible. In these cases we may use other techniques, such as numerical integration, which will be covered in later chapters.

1.4 Response Analysis

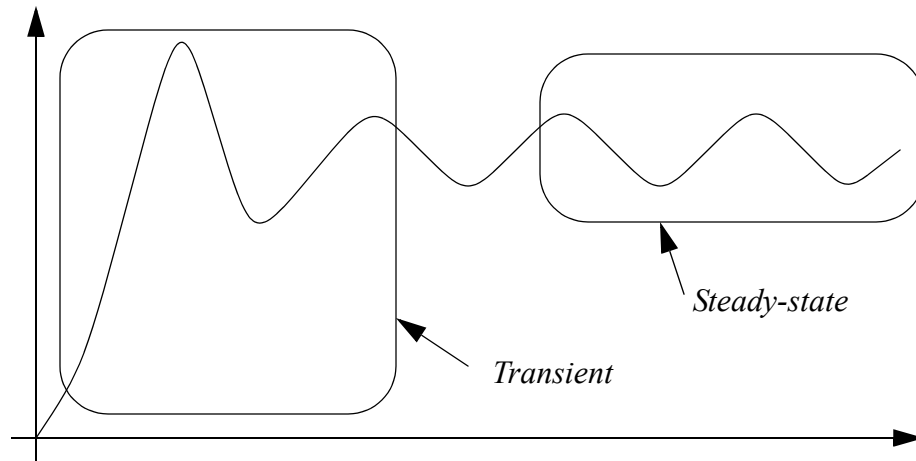
Up to this point we have mostly discussed the process of calculating the system response. As an engineer, obtaining the response is important, but evaluating the results is more important. The most critical design consideration is system stability. In most cases a system should be inherently stable in all situations, such as a car "cruise control". In other

cases an unstable system may be the objective, such as an explosive device. Simple methods for determining the stability of a system are listed below:

1. If a step input causes the system to go to infinity, it will be inherently unstable.
2. A ramp input might cause the system to go to infinity; if this is the case, the system might not respond well to constant change.
3. If the response to a sinusoidal input grows with each cycle, the system is probably resonating, and will become unstable.

Beyond establishing the stability of a system, we must also consider general performance. This includes the time constant for a first-order system, or damping coefficient and natural frequency for a second-order system. For example, assume we have designed an elevator that is a second-order system. If it is under damped the elevator will oscillate, possibly leading to motion sickness, or worse. If the elevator is over damped it will take longer to get to floors. If it is critically damped it will reach the floors quickly, without overshoot.

Engineers distinguish between initial setting effects (transient) and long term effects (steady-state). The transient effects are closely related to the homogeneous solution to the differential equations and the initial conditions. The steady-state effects occur after some period of time when the system is acting in a repeatable or non-changing form. Figure 1.30 shows a system response. The transient effects at the beginning include a quick rise time and an overshoot. The steady-state response settles down to a constant amplitude sine wave.



Note: the transient response is predicted with the homogeneous solution. The steady state response is mainly predicted with the particular solution, although in some cases the homogeneous solution might have steady state effects, such as a non-decaying oscillation.

Figure 1.30 A system response with transient and steady-state effects

1.5 Non-Linear Systems

Non-linear systems cannot be described with a linear differential equation. A basic linear differential equation has coefficients that are constant, and the derivatives are all first order. Examples of non-linear differential equations are shown in Figure 1.31.

$$\dot{x} + x^2 = 5$$

$$\dot{x}^2 + x = 5$$

$$\dot{x} + \log(x) = 5$$

$$\dot{x} + (5t)x = 5$$

Note: the sources of non-linearity are circled.

Figure 1.31 Examples of non-linear differential equations

Examples of system conditions that lead to non-linear solutions are,

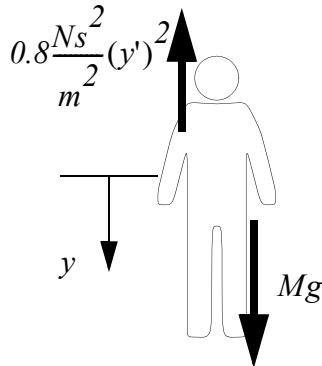
- aerodynamic drag
- forces that are a squared function of distance
- devices with non-linear responses

Explicitly solving non-linear differential equations can be difficult, and will typically involve complex solutions for simple problems.

1.5.1 Non-Linear Differential Equations

A non-linear differential equation is presented in Figure 1.32. It involves a person ejected from an aircraft with a drag force coefficient of 0.8. (Note: This coefficient is calculated using the drag coefficient and other properties such as the speed of sound and cross sectional area.) The FBD shows the sum of forces, and the resulting differential equation. The velocity squared term makes the equation non-linear, and so it cannot be analyzed with the previous methods. In this case the terminal velocity is calculated by setting the acceleration to zero. This results in a maximum speed of 126 kph.

Consider the differential equation for a 100kg human ejected from an airplane. The aerodynamic drag will introduce a squared variable, therefore making the equation non-linear.



$$\sum F_y = 0.8(\dot{y})^2 - Mg = -M\ddot{y}$$

$$100kg\ddot{y} + 0.8\frac{Ns^2}{m^2}(\dot{y})^2 = 100kg9.81\frac{N}{kg}$$

$$100kg\ddot{y} + 0.8\frac{Ns^2}{m^2}(\dot{y})^2 = 981N$$

$$100kg\ddot{y} + 0.8kg\frac{m}{s^2}\frac{s^2}{m^2}(\dot{y})^2 = 981kg\frac{m}{s^2}$$

$$100\ddot{y} + 0.8m^{-1}(\dot{y})^2 = 981ms^{-2}$$

$$\ddot{y} + 8 \times 10^{-3} m^{-1} (\dot{y})^2 = 9.81 ms^{-2}$$

The terminal velocity can be found by setting the acceleration to zero.

$$(0) + 8 \times 10^{-3} m^{-1} (\dot{y})^2 = 9.81 ms^{-2}$$

$$\dot{y} = \sqrt{\frac{9.81 ms^{-2}}{8 \times 10^{-3} m^{-1}}} = \sqrt{\frac{9.81}{8 \times 10^{-3}} m^2 s^{-2}} = 35.0 \frac{m}{s} = 126 \frac{km}{h}$$

Figure 1.32 Development of a non-linear differential equation

The equation can also be solved using explicit integration, as shown in Figure 1.33. In this case the equation is separated and rearranged to isolate the 'v' terms on the left, and time on the right. The term is then integrated in Figure 1.34 and Figure 1.35. The final form of the equation is non-trivial, but contains e-to-t terms, as we would expect.

An explicit solution can begin by replacing the position variable with a velocity variable and rewriting the equation as a separable differential equation.

$$100\ddot{y} + 0.8m^{-1}(\dot{y})^2 = 981ms^{-2}$$

$$100\dot{v} + 0.8m^{-1}v^2 = 981ms^{-2}$$

$$100\frac{dv}{dt} + 0.8m^{-1}v^2 = 981ms^{-2}$$

$$100\frac{dv}{dt} = 981ms^{-2} - 0.8m^{-1}v^2$$

$$\frac{100}{981ms^{-2} - 0.8m^{-1}v^2}dv = dt$$

$$\int \frac{\frac{100}{-0.8m^{-1}}}{\frac{981}{-0.8m^{-1}}ms^{-2} + v^2} dv = \int dt$$

$$\int \frac{-125m}{v^2 - 1226.25m^2s^{-2}} dv = t + C_1$$

$$\int \frac{-125m}{\left(v + 35.02\frac{m}{s}\right)\left(v - 35.02\frac{m}{s}\right)} dv = t + C_1$$

Figure 1.33 Developing an integral

This can be reduced with a partial fraction expansion.

$$\int \left[\frac{A}{\left(v + 35.02 \frac{m}{s}\right)} + \frac{B}{\left(v - 35.02 \frac{m}{s}\right)} \right] dv = t + C_1$$

$$Av - A\left(35.02 \frac{m}{s}\right) + Bv + B\left(35.02 \frac{m}{s}\right) = -125m$$

$$v(A + B) + 35.02 \frac{m}{s}(-A + B) = -125m$$

$$A + B = 0$$

$$A = -B$$

$$35.02 \frac{m}{s}(-A + B) = -125m$$

$$(-(-B) + B) = -\frac{125}{35.02}s$$

$$B = -1.785s$$

$$A = 1.785s$$

$$\int \left[\frac{1.785s}{\left(v + 35.02 \frac{m}{s}\right)} + \frac{-1.785s}{\left(v - 35.02 \frac{m}{s}\right)} \right] dv = t + C_1$$

The integral can then be solved using an identity from the integral table. In this case the integration constants can be left off because they are redundant with the one on the right hand side.

$$1.785s \ln \left| v + 35.02 \frac{m}{s} \right| - 1.785s \ln \left| v - 35.02 \frac{m}{s} \right| = t + C_1$$

$$1.785s \ln \left| \frac{v + 35.02 \frac{m}{s}}{v - 35.02 \frac{m}{s}} \right| = t + C_1$$

$$\int (a + bx)^{-1} dx = \frac{\ln|a + bx|}{b} + C$$

$$\left| \frac{v + 35.02 \frac{m}{s}}{v - 35.02 \frac{m}{s}} \right| = e^{\frac{t}{1.785s} + C_1}$$

$$\left| \frac{v + 35.02 \frac{m}{s}}{v - 35.02 \frac{m}{s}} \right| = e^{C_1} e^{\frac{t}{1.785s}}$$

Figure 1.34 Solution of the integral

$$\left| \frac{v + 35.02 \frac{m}{s}}{v - 35.02 \frac{m}{s}} \right| = C_2 e^{\frac{t}{1.785s}}$$

An initial velocity of zero can be assumed to find the value of the integration constant

$$\left| \frac{0 + 35.02 \frac{m}{s}}{0 - 35.02 \frac{m}{s}} \right| = C_2 e^{\frac{0}{1.785s}} \quad I = C_2$$

This can then be simplified, and the absolute value sign eliminated.

$$\frac{v + 35.02 \frac{m}{s}}{v - 35.02 \frac{m}{s}} = \pm e^{\frac{t}{1.785s}}$$

$$v + 35.02 \frac{m}{s} = \pm v e^{\frac{t}{1.785s}} \mp 35.02 \frac{m}{s} e^{\frac{t}{1.785s}}$$

$$v \left(1 \mp e^{\frac{t}{1.785s}} \right) = \mp 35.02 \frac{m}{s} e^{\frac{t}{1.785s}} - 35.02 \frac{m}{s}$$

$$v = 35.02 \frac{m}{s} \left(\frac{\mp e^{\frac{t}{1.785s}} - 1}{1 \mp e^{\frac{t}{1.785s}}} \right) \quad 0 = 35.02 \frac{m}{s} \left(\frac{\mp I - 1}{I \mp 1} \right) = \left(\frac{1 - I}{1 + I} \right) = \frac{0}{2}$$

$$v = 35.02 \frac{m}{s} \left(\frac{e^{\frac{t}{1.785s}} - 1}{1 + e^{\frac{t}{1.785s}}} \right)$$

Figure 1.35 Solution of the integral and application of the initial conditions

As evident from the example, non-linear equations are involved and don't utilize routine methods. Typically the numerical methods discussed in the next chapter are preferred.

1.5.2 Non-Linear Equation Terms

If our models include a device that is non-linear and we want to use a linear tech-

nique to solve the equation, we will need to linearize the model before we can proceed. A non-linear system can be approximated with a linear equation using the following method.

1. Pick an operating point or range for the component.
2. Find a constant value that relates a change in the input to a change in the output.
3. Develop a linear equation.
4. Use the linear equation for the analysis.

A linearized differential equation can be approximately solved using known techniques as long as the system doesn't travel too far from the linearized point. The example in Figure 1.36 shows the linearization of a non-linear equation about a given operating point. This equation will be approximately correct as long as the first derivative doesn't move too far from 100. When this value does, the new velocity can be calculated.

Assume we have the non-linear differential equation below. It can be solved by linearizing the value about the operating point

Given,

$$\dot{y}^2 + 4y = 200 \quad y(0) = 10$$

We can make the equation linear by replacing the velocity squared term with the velocity times the actual velocity. As long as the system doesn't vary too much from the given velocity the model should be reasonably accurate.

$$\dot{y} = \pm\sqrt{200 - 4y}$$

$$\dot{y}(0) = \pm\sqrt{200 - 4(10)} = \pm 12.65$$

$$12.65\dot{y} + 4y = 20$$

This system may now be solved as a linear differential equation. If the velocity (first derivative of y) changes significantly, then the differential equation should be changed to reflect this.

Homogeneous:

$$12.65\dot{y} + 4y = 0$$

$$12.65A + 4 = 0$$

$$A = -0.316$$

$$y_h = Ce^{-0.316t}$$

Particular:

$$y_p = A$$

$$12.65(0) + 4A = 200$$

$$A = 50$$

Initial conditions:

$$y(t) = Ce^{-0.316t} + 50$$

$$10 = Ce^0 + 50$$

$$C = -40$$

$$y(t) = -40e^{-0.316t} + 50$$

Figure 1.36 Linearizing a differential equation

If the velocity (first derivative of y) changes significantly, then the differential equation should be changed to reflect this. For example we could decide to recalculate the equation value after 0.1s.

$$y(0.1) = -40e^{-0.316(0.1)} + 50 = 11.24$$

$$\frac{d}{dt}y(0.1) = -40(-0.316)e^{-0.316(0.1)} = 12.25 \quad \text{Note: a small change}$$

$$12.25y' + 4y = 20$$

Now recalculate the solution to the differential equation.

Homogeneous:

$$12.25\dot{y} + 4y = 0$$

$$12.25A + 4 = 0 \quad A = -0.327$$

$$y_h = Ce^{-0.327t}$$

Particular:

$$y_p = A$$

$$12.25(0) + 4A = 20 \quad A = 50$$

Initial conditions:

$$y(t) = Ce^{-0.327t} + 50$$

$$11.24 = Ce^{0.1} + 50 \quad C = -35.070575$$

$$y(t) = -35.07e^{-0.316t} + 50$$

Notice that the values have shifted slightly, and as the analysis progresses the equations will adjust slowly. Higher accuracy can be obtained using smaller steps in time.

Figure 1.37 Linearizing a differential equation

1.5.3 Changing Systems

In practical systems, the forces at work are continually changing. For example a system often experiences a static friction force when motion is starting, but once motion starts it is replaced with a smaller kinetic friction. Another example is tension in a cable. When in tension a cable acts as a spring. But, when in compression the force goes to zero.

Consider the example in Figure 1.38. A mass is pulled by a springy cable. The

right hand side of the cable is being pulled at a constant rate, while the block is free to move, only restricted by friction forces and inertia. At the beginning all components are at rest and undeflected.

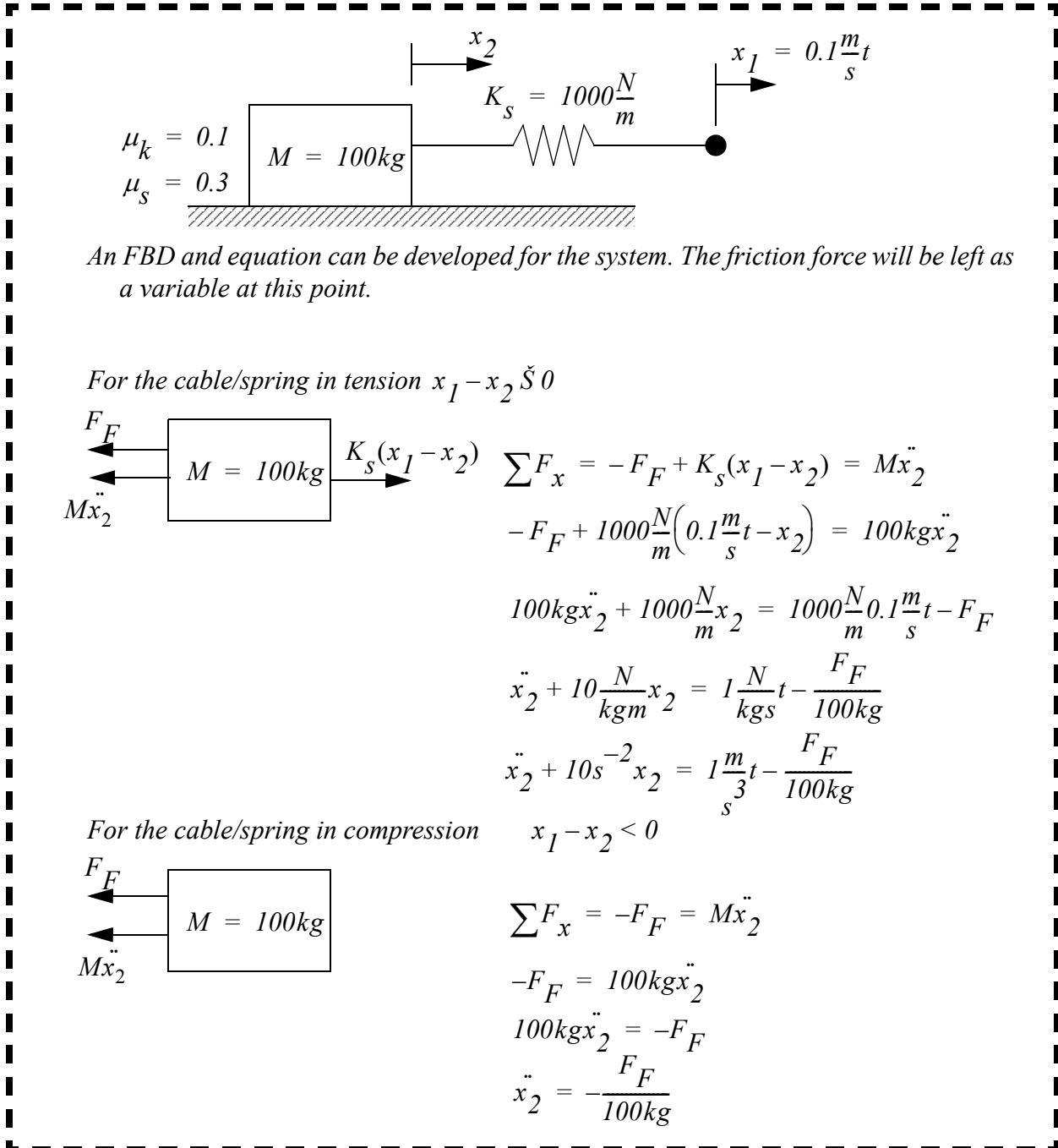


Figure 1.38 A differential equation for a mass pulled by a springy cable

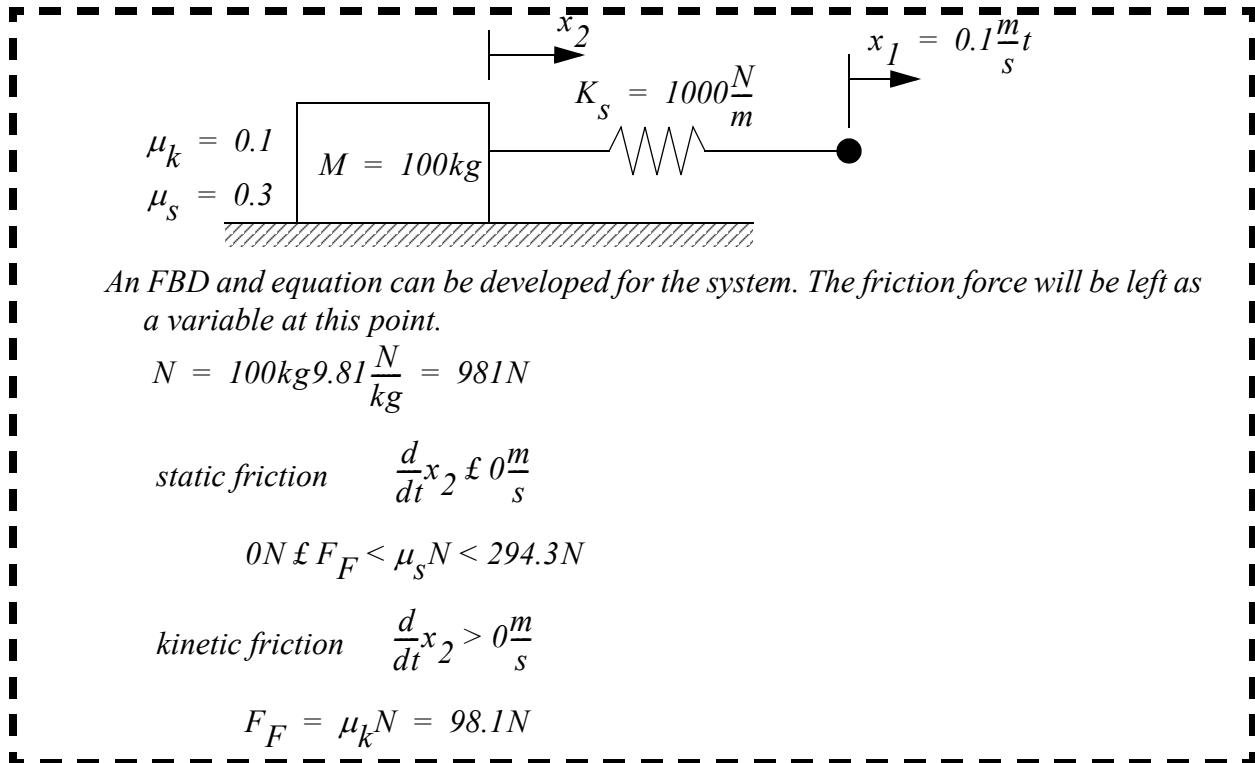


Figure 1.39 Friction forces for the mass

The analysis of the system begins by assuming the system starts at rest and undeflected. In this case the cable/spring will be undeflected with no force, and the mass will be experiencing static friction. Therefore the block will stay in place until the cable stretches enough to overcome the static friction.

$$x_2 = 0 \quad \ddot{x}_2 = 0 \quad F_F = 294.3\text{N}$$

$$\ddot{x}_2 + 10\text{s}^{-2}x_2 = 1 \frac{\text{m}}{\text{s}^3}t - \frac{F_F}{100\text{kg}}$$

$$0 + 10\text{s}^{-2} \cdot 0 = 1 \frac{\text{m}}{\text{s}^3}t - \frac{294.3\text{N}}{100\text{kg}}$$

$$1 \frac{\text{m}}{\text{s}^3}t = \frac{294.3\text{kgm}}{100\text{kgs}^2}$$

$$t = 2.943\text{s}$$

Therefore the system is static from 0 to 2.943s

Figure 1.40 Analysis of the object before motion begins

After motion begins the object will only experience kinetic friction, and continue to accelerate until the cable/spring becomes loose in compression. This stage of motion requires the solution of a differential equation.

$$\ddot{x}_2 + 10s^{-2}x_2 = 1\frac{m}{3}t - \frac{98.1N}{100kg}$$

For the homogeneous,

$$\ddot{x}_2 + 10s^{-2}x_2 = 0$$

$$A + 10s^{-2} = 0 \quad A = \pm 3.16js^{-1}$$

$$x_h = C_1 \sin(3.16t + C_2)$$

For the particular,

$$x_p = At + B \quad \dot{x}_p = A \quad \ddot{x}_p = 0$$

$$0 + 10s^{-2}(At + B) = 1\frac{m}{3}t - \frac{98.1N}{100kg}$$

$$10s^{-2}A = 1\frac{m}{3} \quad A = 0.1\frac{m}{s}$$

$$10s^{-2}B = -\frac{98.1N}{100kg} \quad B = -0.0981m$$

Figure 1.41 Analysis of the object after motion begins

For the initial conditions,

$$x(2.943s) = 0m \quad \frac{d}{dt}x(2.943s) = 0\frac{m}{s}$$

$$x(t) = C_1 \sin(3.16t + C_2) + 0.1\frac{m}{s}t - 0.0981m$$

$$0 = C_1 \sin(3.16(2.943s) + C_2) + 0.1\frac{m}{s}(2.943s) - 0.0981m$$

$$C_1 \sin(9.29988 + C_2) = -0.1962$$

$$\frac{d}{dt}x(t) = 3.16C_1 \cos(3.16t + C_2) + 0.1\frac{m}{s}$$

$$0 = 3.16C_1 \cos(3.16(2.943) + C_2) + 0.1\frac{m}{s}$$

$$C_1 \cos(9.29988 + C_2) = -0.0316$$

$$\frac{C_1 \sin(9.29988 + C_2)}{C_1 \cos(9.29988 + C_2)} = \frac{-0.1962}{-0.0316}$$

$$\tan(9.29988 + C_2) = 6.209 \quad C_2 = (-7.889 + \pi n)\text{rad} \quad n \in \mathbb{I}$$

$$C_1 = \frac{-0.1962}{\sin(9.29988 - 7.889)} = -0.199m$$

$$x(t) = -0.199m \sin(3.16t - 7.889\text{rad}) + 0.1\frac{m}{s}t - 0.0981m$$

$$\frac{d}{dt}x(t) = -0.199(3.16)m \cos(3.16t - 7.889\text{rad}) + 0.1\frac{m}{s}$$

Figure 1.42 Analysis of the object after motion begins

The equation of motion changes after the cable becomes slack. This point in time can be determined when the displacement of the block equals the displacement of the cable/spring end.

$$0.1 \frac{m}{s} t = -0.199 m \sin(3.16t - 7.889 \text{ rad}) + 0.1 \frac{m}{s} t - 0.0981 m$$

$$-0.199 m \sin(3.16t - 7.889 \text{ rad}) = 0.0981 m$$

$$3.16t - 7.889 + \pi n = -0.51549413 \quad t = 3.328 s$$

$$x(3.328) = 0.137 m \quad \frac{d}{dt}x(3.328) = 0.648 \frac{m}{s}$$

After this the differential equation without the cable/spring is used.

$$x_2'' = \frac{-98.1 N}{100 \text{ kg}} = -0.981 \frac{m}{s^2}$$

$$\dot{x}_2 = \left(-0.981 \frac{m}{s^2}\right)t + C_1$$

$$0.648 \frac{m}{s} = \left(-0.981 \frac{m}{s^2}\right)(3.328 s) + C_1$$

$$C_1 = 3.913 \frac{m}{s}$$

$$x_2 = \left(-\frac{0.981 m}{2 s^2}\right)t^2 + 3.913 \frac{m}{s}t + C_2$$

$$0.137 m = \left(-\frac{0.981 m}{2 s^2}\right)(3.328 s)^2 + 3.913 \frac{m}{s}(3.328 s) + C_2$$

$$C_2 = -7.453 m$$

$$x_2(t) = \left(-\frac{0.981 m}{2 s^2}\right)t^2 + 3.913 \frac{m}{s}t - 7.453 m$$

This motion continues until the block stops moving.

$$0 = \left(-0.981 \frac{m}{s^2}\right)t + 3.913 \frac{m}{s}$$

$$t = 3.989 s$$

The solution can continue, considering when to switch the analysis conditions.

Figure 1.43 Determining when the cable become slack

1.6 Case Study

A typical vibration control system design is described in Figure 1.44.

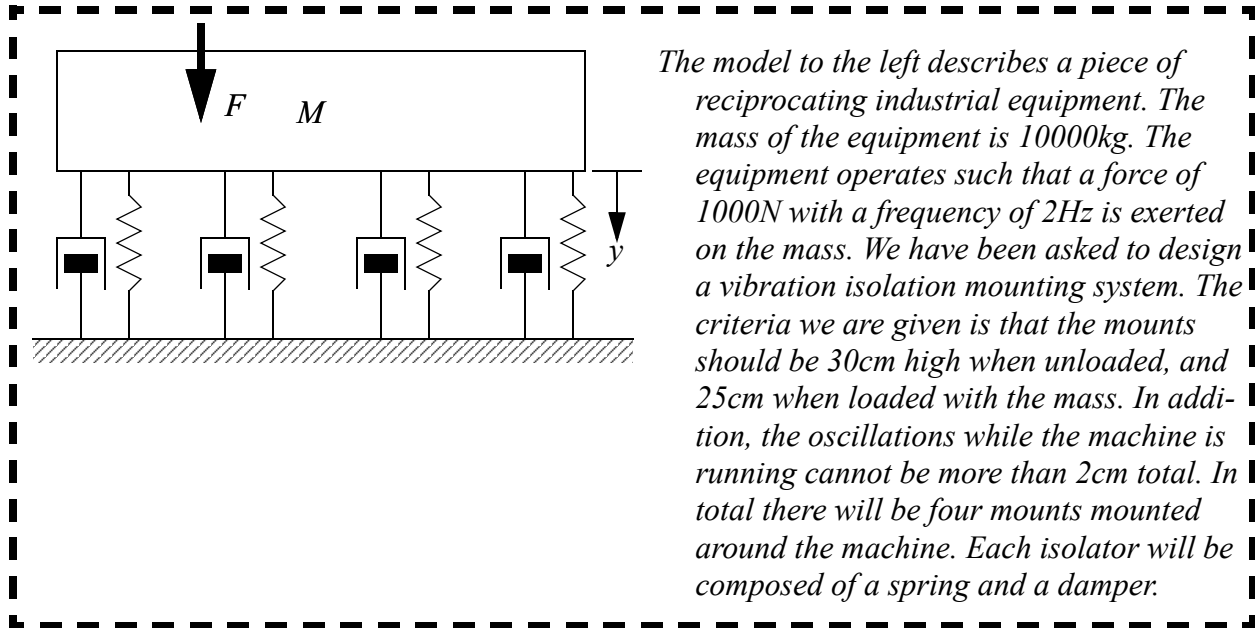


Figure 1.44 A vibration control system

There are a number of elements to the design and analysis of this system, but as usual the best place to begin is by developing a free body diagram, and a differential equation. This is done in Figure 1.45.

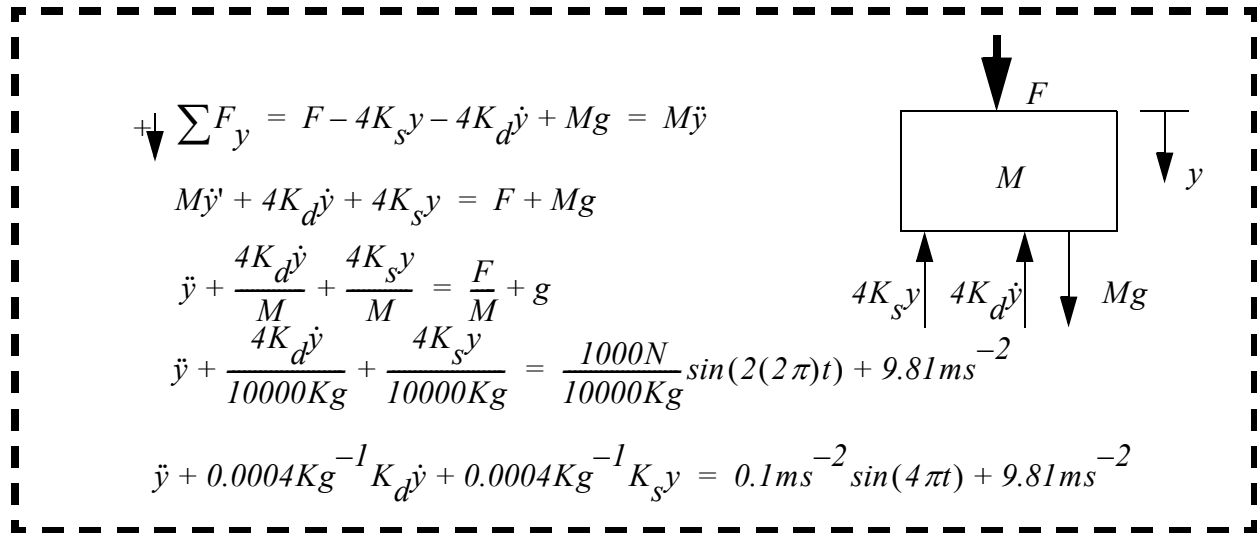


Figure 1.45 FBD and derivation of equation

Using the differential equation, the spring values can be found by assuming the machine is at rest. This is done in Figure 1.46.

When the system is at rest the equation is simplified; the acceleration and velocity terms both become zero. In addition, we will assume that the cyclic force is not applied for the unloaded/loaded case. This simplifies the differential equation by eliminating several terms.

$$0.0004\text{Kg}^{-1} K_s y = 9.81\text{ms}^{-2}$$

Now we can consider that when unloaded the spring is 0.30m long, and after loading the spring is 0.25m long. This will result in a downward compression of 0.05m, in the positive y direction.

$$0.0004\text{Kg}^{-1} K_s (0.05\text{m}) = 9.81\text{ms}^{-2}$$

$$K_s = \frac{9.81}{0.0004(0.05)} \text{Kgms}^{-2} \text{m}^{-1}$$

$$\therefore K_s = 491\text{KNm}^{-1}$$

Figure 1.46 Calculation of the spring coefficient

The remaining unknown is the damping coefficient. At this point we have determined the range of motion of the mass. This can be done by developing the particular

solution of the differential equation, as it will contain the steady-state oscillations caused by the forces as shown in Figure 1.47.

$$\ddot{y} + 0.0004Kg^{-1}K_d\dot{y} + 0.0004Kg^{-1}(491KNm^{-1})y = 0.1ms^{-2}\sin(4\pi t) + 9.81ms^{-2}$$

$$\ddot{y} + 0.0004Kg^{-1}K_d\dot{y} + 196s^{-2}y = 0.1ms^{-2}\sin(4\pi t) + 9.81ms^{-2}$$

The particular solution can now be found by guessing a value, and solving for the coefficients. (Note: The units in the expression are uniform (i.e., the same in each term) and will be omitted for brevity.)

$$y = A\sin(4\pi t) + B\cos(4\pi t) + C$$

$$y' = 4\pi A\cos(4\pi t) - 4\pi B\sin(4\pi t)$$

$$y'' = -16\pi^2 A\sin(4\pi t) - 16\pi^2 B\cos(4\pi t)$$

$$\therefore (-16\pi^2 A\sin(4\pi t) - 16\pi^2 B\cos(4\pi t)) + 0.0004K_d(4\pi A\cos(4\pi t) - 4\pi B\sin(4\pi t)) + 196(A\sin(4\pi t) + B\cos(4\pi t) + C) = 0.1\sin(4\pi t) + 9.81$$

$$-16\pi^2 B + 0.0004K_d 4\pi A + 196A = 0$$

$$B = A(31.8 \times 10^{-6}K_d + 1.24)$$

$$-16\pi^2 A + 0.0004K_d(-4\pi B) + 196A = 0.1$$

$$A(-16\pi^2 + 196) + B(-5.0 \times 10^{-3}K_d) = 0.1$$

$$A(-16\pi^2 + 196) + A(31.8 \times 10^{-6}K_d + 1.24)(-5.0 \times 10^{-3}K_d) = 0.1$$

$$A = \frac{0.1}{-16\pi^2 + 196 + (31.8 \times 10^{-6}K_d + 1.24)(-5.0 \times 10^{-3}K_d)}$$

$$A = \frac{0.1}{K_d^2(-159 \times 10^{-9}) + K_d(-6.2 \times 10^{-3}) + 38.1}$$

$$B = \frac{3.18 \times 10^{-6}K_d - 0.124}{K_d^2(-159 \times 10^{-9}) + K_d(-6.2 \times 10^{-3}) + 38.1}$$

$$C = 9.81ms^{-2}$$

Figure 1.47 Particular solution of the differential equation

The particular solution can be used to find a damping coefficient that will give an overall oscillation of 0.02m, as shown in Figure 1.48. In this case Mathcad was used to find the solution, although it could have also been found by factoring out the algebra, and finding the roots of the resulting polynomial.

In the previous particular solution the values were split into cosine and sine components. The magnitude of oscillation can be calculated with the Pythagorean formula.

$$\text{magnitude} = \sqrt{A^2 + B^2}$$

$$\text{magnitude} = \frac{\sqrt{(0.1)^2 + \left((3.18 \cdot 10^{-6})K_d - 0.124\right)^2}}{K_d^2(-159 \cdot 10^{-9}) + K_d(-6.2 \cdot 10^{-3}) + 38.1}$$

The design requirements call for a maximum oscillation of 0.02m, or a magnitude of 0.01m.

$$0.01 = \frac{\sqrt{(0.1)^2 + \left((3.18 \cdot 10^{-6})K_d - 0.124\right)^2}}{K_d^2(-159 \cdot 10^{-9}) + K_d(-6.2 \cdot 10^{-3}) + 38.1}$$

A given-find block was used in Mathcad to obtain a damper value of,

$$K_d = 3411 \frac{\text{N}\cdot\text{s}}{\text{m}}$$

Aside: the Mathcad solution

$$f(k) := \frac{\sqrt{0.01 + \left[(3.18 \cdot 10^{-6} \cdot k) - 0.124\right]^2}}{\left[k \cdot k \cdot (-159 \cdot 10^{-9})\right] + k \cdot (-6.2 \cdot 10^{-3}) + 38.1}$$

$$k_d := 1$$

given

$$f(k_d) = 0.01$$

$$\text{find}(k_d) = 3.411 \times 10^3$$

Figure 1.48 Determining the damping coefficient

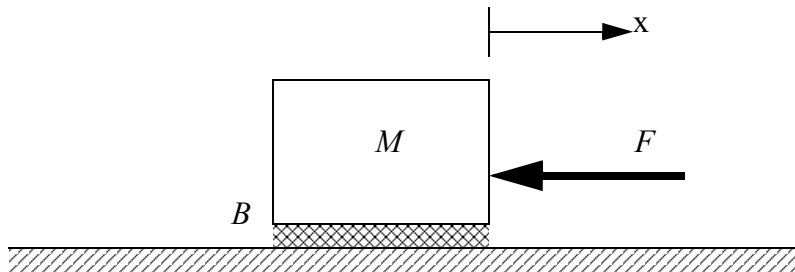
The values of the spring and damping coefficients can be used to select actual components. Some companies will design and build their own components. Components can also be acquired by searching catalogs, or requesting custom designs from other companies.

1.7 Summary

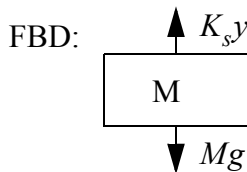
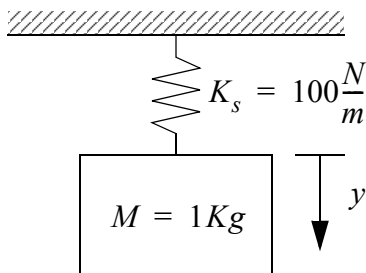
- First and second-order differential equations were analyzed explicitly.
- First and second-order responses were examined.
- The topic of analysis was discussed.
- A case study looked at a second-order system.
- Non-linear systems can be analyzed by making them linear.

1.8 Problems

1. The mass, M , illustrated below starts at rest. It can slide across a surface, but the motion is opposed by viscous friction (damping) with the coefficient B . Initially the system starts at rest, when a constant force, F , is applied. Write the differential equation for the mass, and solve the differential equation. Leave the results in variable form.



2. The following differential equation was derived for a mass suspended with a spring. At time $0s$ the system is released and allowed to drop. It then oscillates. Solve the differential equation to find the motion as a function of time.



$$\begin{aligned}
 \uparrow \sum F_y &= K_s y - Mg = -M\ddot{y} \\
 \left(100 \frac{N}{m}\right)y - (1Kg)\left(9.81 \frac{N}{Kg}\right) &= (-1Kg)\ddot{y} \\
 \left(1 \frac{Nm}{s^2}\right)\ddot{y} + \left(100 \frac{N}{m}\right)y &= 9.81 N \\
 (1Kg)\ddot{y} + \left(100 \frac{Kgm}{ms^2}\right)y &= 9.81 \frac{Kgm}{s^2} \\
 \ddot{y} + (100s^{-2})y &= 9.81ms^{-2} \\
 y_0 = 0m \quad \dot{y}_0 &= 0ms^{-1}
 \end{aligned}$$

3. Solve the following differential equation with the three given cases. All of the systems have a

step input 'y' and start undeflected and at rest.

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2x = y \quad \text{initial conditions} \quad \dot{x} = 0$$

$$x = 0$$

$$y = 1$$

case 1: $\xi = 0.5 \quad \omega_n = 10$

case 2: $\xi = 1 \quad \omega_n = 10$

case 3: $\xi = 2 \quad \omega_n = 10$

4. Solve the following differential equation with the given initial conditions and draw a sketch of the first 5 seconds. The input is a step function that turns on at t=0.

$$0.5\ddot{V}_o + 0.6\dot{V}_o + 2.1V_o = 3V_i + 2 \quad \text{initial conditions} \quad V_i = 5$$

$$V_o = 0$$

$$\dot{V}_o = 0$$

5. Solve the following differential equation with the given initial conditions and draw a sketch of the first 5 seconds. The input is a step function that turns on at t=0.

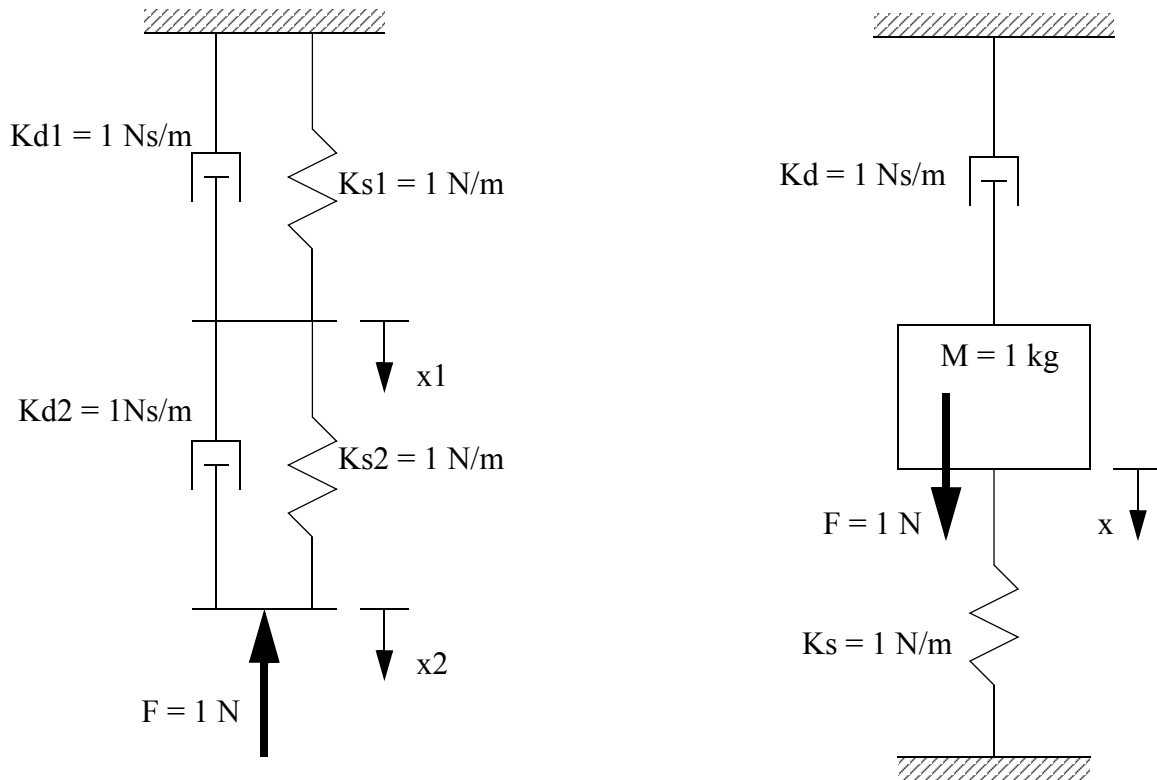
$$0.5\ddot{V}_o + 0.6\dot{V}_o + 2.1V_o = 3V_i + 2 \quad \text{initial conditions} \quad V_i = 5$$

$$V_o = 0$$

$$\dot{V}_o = 1$$

6. a) Write the differential equations for the system below. Solve the equations for x assuming that

the system is at rest and undeflected before $t=0$. Also assume that gravity is present.



b) State whether each system is first or second-order. If the system is first-order find the time constant. If it is second-order find the natural frequency and damping ratio.

7. Solve the following differential equation with the three given cases. All of the systems have a sinusoidal input 'y' and start undeflected and at rest.

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2x = y$$

initial conditions $\dot{x} = 0$

$x = 0$

$y = \sin(t)$

case 1: $\xi = 0.5$ $\omega_n = 10$

case 2: $\xi = 1$ $\omega_n = 10$

case 3: $\xi = 2$ $\omega_n = 10$

8. A spring damper system supports a mass of 34N. If it has a spring constant of 20.6N/cm, what is the systems natural frequency?

9. Using a standard lumped parameter model the weight is 36N, stiffness is $2.06 \cdot 10^3 \text{ N/m}$ and damping is 100Ns/m. What are the natural frequency (Hz) and damping ratio?

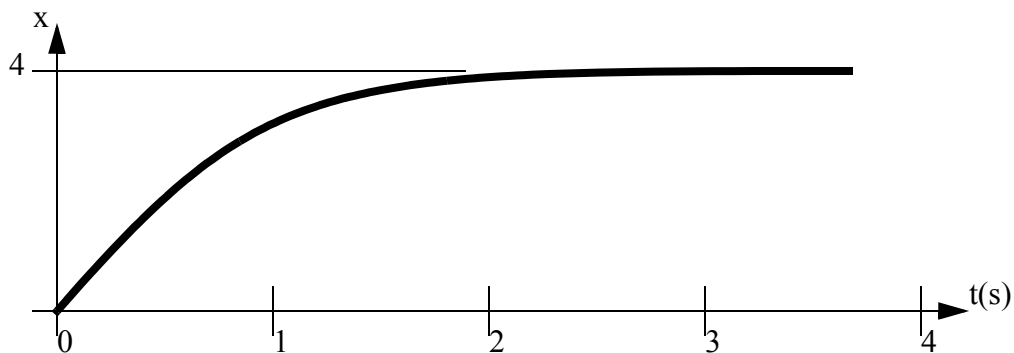
10. What is the differential equation for a second-order system that responds to a step input with an overshoot of 20%, with a delay of 0.4 seconds to the first peak?
11. A system is to be approximated with a mass-spring-damper model using the following parameters: weight 28N, viscous damping 6Ns/m, and stiffness 36N/m. Calculate the undamped natural frequency (Hz) of the system, the damping ratio and describe the type of response you would expect if the mass were displaced and released. What additional damping would be required to make the system critically damped?

$$M\ddot{x} + K_d\dot{x} + K_s x = F$$

12. Solve the differential equation below using homogeneous and particular solutions. Assume the system starts undeflected and at rest.

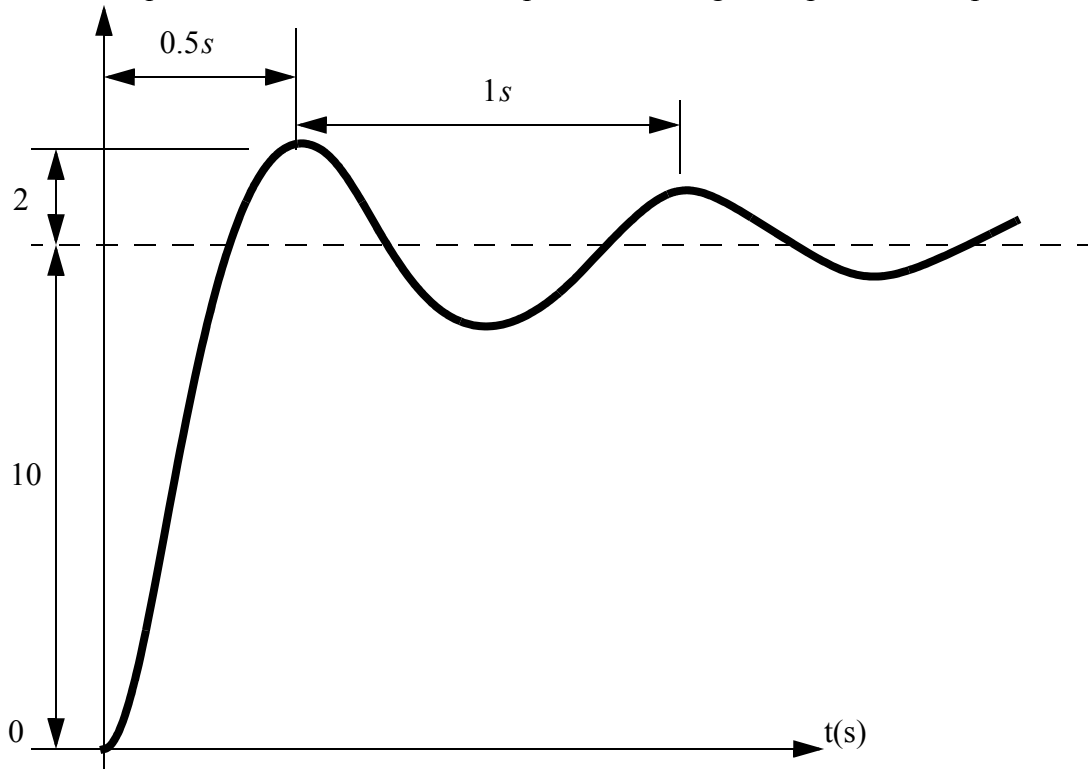
$$\ddot{\theta} + 40\dot{\theta} + 20\theta = 4$$

13. What would the displacement amplitude after 100ms for a system having a natural frequency of 13 rads/sec and a damping ratio of 0.20. Assume an initial displacement of 50mm, and a steady state displacement of 0mm. (Hint: Find the response as a function of time.)
14. Determine the first order differential equation given the graphical response shown below. Assume the input is a step function.

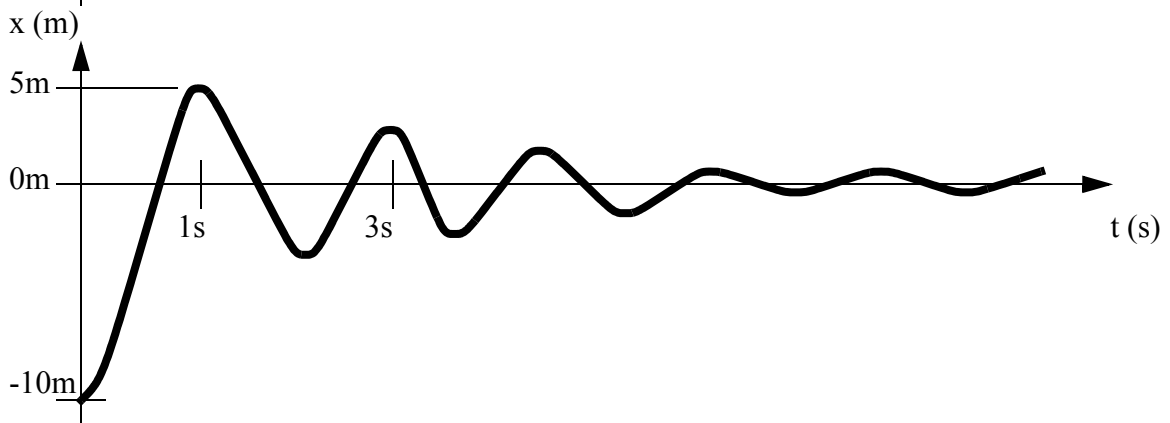
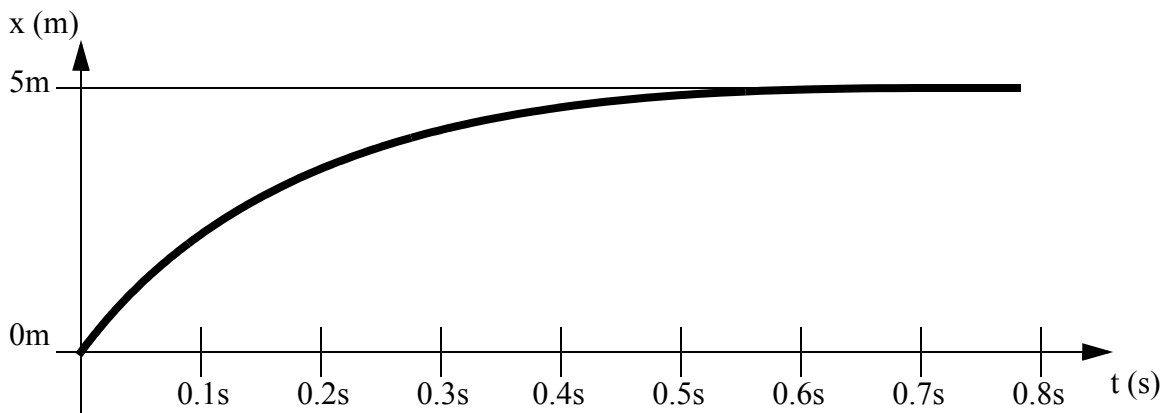


15. Explain with graphs how to develop first and second-order equations using experimental data.
16. The second order response below was obtained experimentally. Determine the parameters of

the differential equation that resulted in the response assuming the input was a step function.



17. Develop equations (function of time) for the first and second order responses shown below.



18. A mass-spring-damper system has a mass of 10Kg and a spring coefficient of 1KN/m. Select a damping coefficient so that the system will have an overshoot of 20% for a step input.

1.9 Problems Solutions

1.

$$x(t) = \frac{-FM}{B^2} e^{-\frac{B}{M}t} - \frac{F}{B}t + \frac{FM}{B^2}$$

2.

homogeneous: guess $y_h = e^{At}$ $\dot{y}_h = Ae^{At}$ $\ddot{y}_h = A^2 e^{At}$

$$A^2 e^{At} + (100s^{-2})e^{At} = 0$$

$$A^2 = -100s^{-2} \quad A = \pm 10js^{-1}$$

$$y_h = C_1 \cos(10t + C_2)$$

particular: guess $y_p = A$ $\dot{y}_p = 0$ $\ddot{y}_p = 0$

$$(0) + (100s^{-2})A = 9.81ms^{-2}$$

$$(100s^{-2})A = 9.81ms^{-2}$$

$$A = \frac{9.81ms^{-2}}{100s^{-2}} = 0.0981m$$

$$y_p = 0.0981m$$

initial conditions: $y = y_h + y_p = C_1 \cos(10t + C_2) + 0.0981m$

$$y' = -10C_1 \sin(10t + C_2)$$

for $d/dt y_0 = 0m$:

$$0 = -10C_1 \sin(10(0) + C_2) \quad C_2 = 0$$

for $y_0 = 0m$:

$$0 = C_1 \cos(10(0) + (0)) + 0.0981m$$

$$-0.0981m = C_1 \cos(0) \quad C_1 = -0.0981m$$

$y(t) = (-0.0981m)\cos(10t) + 0.0981m$

3.

case 1: $x(t) = -0.0115e^{-5t} \cos(8.66t - 0.524) + 0.010$

case 2: $x(t) = -0.010e^{-10t} - 0.10te^{-10t} + 0.010$

case 3: $x(t) = 775 \cdot 10^{-6} e^{-37.32t} - 0.0108e^{-2.679t} + 0.010$

4.

$$V_o(t) = -8.465e^{-0.6t} \sin(1.960t + 1.274) + 8.095$$

or

$$V_o(t) = -8.465e^{-0.6t} \cos(1.960t - 0.2971) + 8.095$$

5.

$$V_0(t) = -8.331e^{-0.6t} \cos(1.96t - 0.238) + 8.095$$

6.

a)

$$x_1(t) = e^{-t} - 1$$

$$x_2(t) = 2e^{-t} - 2$$

$$\tau = 1$$

b)

$$x(t) = -12.485e^{-0.5t} \cos(0.866t - 0.524) + 10.81$$

$$\zeta = 0.5 \quad \omega_n = 1$$

7.

case 1: $x(t) = -0.00117e^{-5t} \sin(8.66t - 1.061) + 0.0101 \sin(t - 0.101)$

case 2: $x(t) = (1.96 \cdot 10^{-3})e^{-10t} + (9.9 \cdot 10^{-3})te^{-10t} + (9.9 \cdot 10^{-3})\sin(t + 0.20)$

case 3: $x(t) = (3.5 \cdot 10^{-3})e^{-2.679t} - (18 \cdot 10^{-6})e^{-37.32t} + (9.4 \cdot 10^{-3})\sin(t + 0.382)$

8. 24.37 rad/sec

9. $f_n=3.77\text{Hz}$, $\text{damp}=.575$

10.

$$\ddot{x} + 8.048\dot{x} + 77.88x = F(t)$$

11.

Given $K_d = 6 \frac{N \cdot s}{m}$ $K_s = 36 \frac{N}{m}$ $M = \frac{28 N}{9.81 \frac{N}{kg}} = 2.85 kg$

The typical transfer function for a mass-spring-damper systems is,

$$\ddot{x} + \dot{x} \left(\frac{K_d}{M} \right) + x \left(\frac{K_s}{M} \right) = \frac{F}{M}$$

The second order parameters can be calculated from this.

$$\ddot{x} + \dot{x}(2\zeta\omega_n) + x(\omega_n^2) = y(t)$$

$$\omega_n = \sqrt{\frac{K_s}{M}} = \sqrt{\frac{36 \frac{N}{m}}{2.85 kg}} = \sqrt{\frac{36 \frac{kgm}{ms^2}}{2.85 kg}} = \sqrt{12.63 s^{-2}} = 3.55 \frac{rad}{s} = 0.6 Hz$$

$$\xi = \frac{\left(\frac{K_d}{M} \right)}{2\omega_n} = \frac{6 \frac{N \cdot s}{m}}{2(3.55) \frac{rad}{s} 2.85 kg} = 0.296$$

$$\omega_d = \omega_n \sqrt{1 - \xi^2} = 3.39 \frac{rad}{s}$$

If pulled and released the system would have a decaying oscillation about 0.52Hz

A critically damped system would require a damping coefficient of...

$$\xi = \frac{\left(\frac{K_d}{M} \right)}{2\omega_n} = \frac{K_d}{2(3.55) \frac{rad}{s} 2.85 kg} = 1.00 \quad K_d = 20.2 \frac{Ns}{m}$$

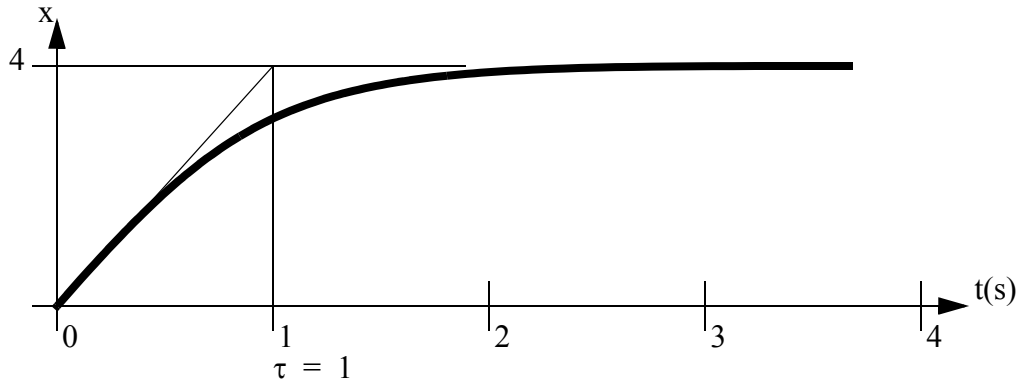
12.

$$\theta(t) = -66 \cdot 10^{-6} e^{-39.50t} - 3.216 e^{0.1383t} + 1.216 e^{-0.3368t} + 2.00$$

13.

$$y(t) = 0.05 e^{-2.6t} \cos(12.74t) = 0.05 e^{-2.6t} \sin\left(12.74t + \frac{\pi}{2}\right)$$

14.



Given the equation form,

$$\dot{x} + \frac{1}{\tau}x = A$$

The values at steady state will be

$$\dot{x} = 0 \quad x = 4$$

So the unknown 'A' can be calculated.

$$0 + \frac{1}{1}4 = A \quad A = 4$$

$$\dot{x} + \frac{1}{1}x = 4$$

$$\dot{x} + x = 4$$

15.

Key points:

First-order:

find initial final values

find time constant with 63% or by slope

use these in standard equation

Second-order:

find damped frequency from graph

find time to first peak

use these in cosine equation

16.

For the first peak:

$$\frac{b}{\Delta x} = e^{-\sigma t_p}$$

$$\frac{2}{10} = e^{-\sigma 0.5}$$

$$\ln\left(\frac{2}{10}\right) = -\sigma 0.5$$

$$\sigma = -2 \ln\left(\frac{2}{10}\right) = 3.219$$

$$\omega_d \approx \frac{\pi}{t_p}$$

For the damped frequency:

$$\omega_d = \frac{2\pi}{1s} = 2\pi$$

$$\xi = \frac{1}{\sqrt{\left(\frac{\pi}{t_p \sigma}\right)^2 + 1}}$$

These values can be used to find the damping coefficient and natural frequency

$$\sigma = \xi \omega_n \quad \omega_n = \frac{3.219}{\xi}$$

$$\omega_d = \omega_n \sqrt{1 - \xi^2}$$

$$2\pi = \frac{3.219}{\xi} \sqrt{1 - \xi^2}$$

$$\left(\frac{2\pi}{3.219}\right)^2 = \frac{1 - \xi^2}{\xi^2}$$

$$\left(\frac{2\pi}{3.219}\right)^2 + 1 = \frac{1}{\xi^2} \quad \xi = \frac{1}{\sqrt{\left(\frac{2\pi}{3.219}\right)^2 + 1}} = 0.4560$$

$$\omega_n = \frac{3.219}{\xi} = \frac{3.219}{0.4560} = 7.059$$

This leads to the final equation using the steady state value of 10

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2x = F$$

$$\ddot{x} + 2(0.4560)(7.059)\dot{x} + (7.059)^2x = F$$

$$\ddot{x} + 6.438\dot{x} + 49.83x = F$$

$$(0) + 6.438(0) + 49.83(10) = F$$

$$F = 498.3$$

$$\ddot{x} + 6.438\dot{x} + 49.83x = 498.3$$

17.

$$x(t) = 5 \left(1 - e^{\frac{-t}{0.18}} \right)$$

$$x(t) = -10e^{-0.693t} \cos(\pi t)$$

1.10 Challenge Problems

1. Write a Scilab program to solve first and second order differential equations for step inputs. The program should accept coefficients for the differential equations and initial conditions. It should then produce a function of time.