38. CALCULUS

Topics:

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Objectives:

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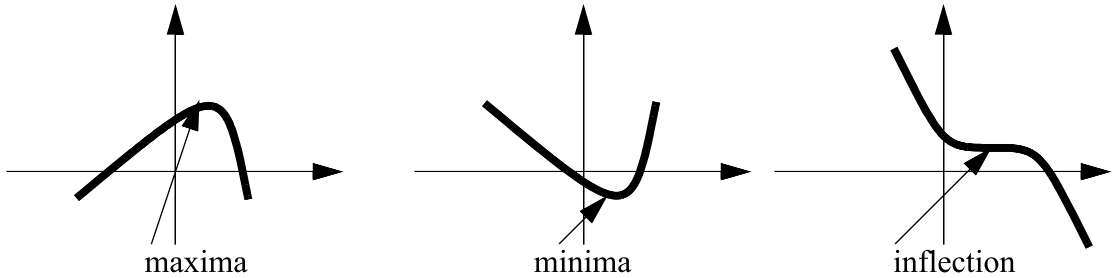
38.1 Introduction

38.2 Derivatives

• The basic definition of a derivative is,

$$\frac{d}{dt}f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

• First derivatives are often used to get the slope of a function. When this is zero the function may be at a minimum/maximum.



• Notations,

$$\frac{d}{dx}y = y'$$

$$\frac{d}{dt}y = \dot{y}$$

• The basic principles of differentiation are, Both u, v and w are functions of x, but this is not shown for brevity.

Also note that C is used as a constant, and all angles are in radians.

$$\frac{d}{dx}(C) = 0$$

$$\frac{d}{dx}(Cu) = (C)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(u + v + \ldots) = \frac{d}{dx}(u) + \frac{d}{dx}(v) + \ldots$$

$$\frac{d}{dx}(u^n) = (nu^{n-1})\frac{d}{dx}(u)$$

$$\frac{d}{dx}(uv) = (u)\frac{d}{dx}(v) + (v)\frac{d}{dx}(u)$$ product rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \left(\frac{v}{v^2}\right)\frac{d}{dx}(u) - \left(\frac{u}{v^2}\right)\frac{d}{dx}(v) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

$$\frac{d}{dx}(uvw) = (uv)\frac{d}{dx}(w) + (uw)\frac{d}{dx}(v) + (vw)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(y) = \frac{d}{du}(y)\frac{d}{dx}(u)$$ chain rule

$$\frac{d}{dx}(u) = \frac{1}{\frac{d}{du}(x)}$$

$$\frac{d}{dx}(y) = \frac{\frac{d}{du}(y)}{\frac{d}{du}(x)}$$

• Examples,

$$\frac{d}{dt}(t^2 + t + 3) = 2t + 1$$

$$\frac{d}{dt}\left(\frac{t^2 + t + 3}{t + 2}\right) = \frac{2t + 1}{t + 2} - \frac{t^2 + t + 3}{(t + 2)^2}$$

• Differentiation rules specific to basic trigonometry and logarithm functions

$$\frac{d}{dx}(\sin u) = (\cos u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\cos u) = (-\sin u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\tan u) = \left(\frac{1}{\cos u}\right)^2\frac{d}{dx}(u)$$

$$\frac{d}{dx}(e^u) = (e^u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\cot u) = (-\csc u)^2\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\sec u) = (\tan u \sec u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\csc u) = (-\csc u \cot u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\sinh u) = (\cosh u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\cosh u) = (\sinh u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\tanh u) = (\operatorname{sech} u)^2\frac{d}{dx}(u)$$

• L'Hospital's rule can be used when evaluating limits that go to infinity.

$$\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x \to a}\left(\frac{\left(\frac{d}{dt}\right)f(x)}{\left(\frac{d}{dt}\right)g(x)}\right) = \lim_{x \to a}\left(\frac{\left(\frac{d}{dt}\right)^2 f(x)}{\left(\frac{d}{dt}\right)^2 g(x)}\right) = \dots$$

• Some techniques used for finding derivatives are, Leibnitz's Rule, (notice the form is similar to the binomial equation) can be used for finding the derivatives of multiplied functions.

$$\left(\frac{d}{dx}\right)^n(uv) = \left(\frac{d}{dx}\right)^0(u)\left(\frac{d}{dx}\right)^n(v) + \binom{n}{1}\left(\frac{d}{dx}\right)^1(u)\left(\frac{d}{dx}\right)^{n-1}(v)$$

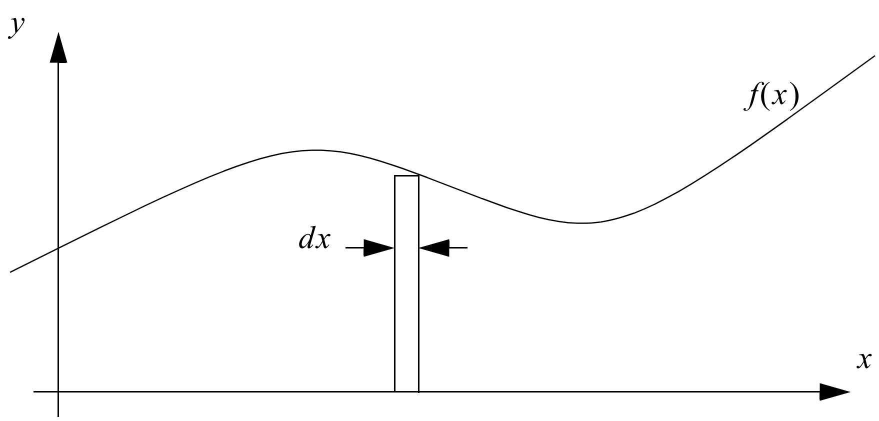
$$\binom{n}{2}\left(\frac{d}{dx}\right)^2(u)\left(\frac{d}{dx}\right)^{n-2}(v) + \ldots + \binom{n}{n}\left(\frac{d}{dx}\right)^n(u)\left(\frac{d}{dx}\right)^0(v)$$

38.3 Integrals

• Integrals are often referred to as anti-derivatives

• definite integrals have boundaries defines. Indefinitie integrals do not have boundaries defines and a constant must be added to the result.

• To set up integrals use integration elements (aka slices),



$$dA = width \cdot height = dx \cdot f(x)$$

$$A = \int f(x)dx$$

• Some basic properties of indefinite integrals (no given start and end limits) include, In the following expressions, u, v, and w are functions of x. in addition to this, C is a constant. and all angles are radians.

$$\int C dx = ax + C$$

$$\int Cf(x)dx = C\int f(x)dx$$

$$\int (u + v + w + \dots)dx = \int u dx + \int v dx + \int w dx + \dots$$

$$\int u dv = uv - \int v du = \text{ integration by parts}$$

$$\int f(Cx)dx = \frac{1}{C}\int f(u)du \qquad u = Cx$$

$$\int F(f(x))dx = \int F(u)\frac{d}{du}(x)du = \int \frac{F(u)}{f'(x)}du \qquad u = f(x)$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \int \frac{1}{x}dx = \ln|x| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \qquad \int e^x dx = e^x + C$$

• Some of the trigonometric integrals are,

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int (\sin x)^2 dx = -\frac{\sin x \cos x + x}{2} + C$$

$$\int (\cos x)^2 dx = \frac{\sin x \cos x + x}{2} + C$$

$$\int (\sin x)^3 dx = -\frac{\cos x((\sin x)^2 + 2)}{3} + C$$

$$\int (\cos x)^3 dx = \frac{\sin x((\cos x)^2 + 2)}{3} + C$$

$$\int (\cos x)^4 dx = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

$$\int \cos x(\sin x)^n dx = \frac{(\sin x)^{n+1}}{n+1} + C$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$

$$\int \tanh x\,dx = \ln(\cosh x) + C$$

$$\int x\cos(ax)dx = \frac{\cos(ax)}{a^2} + \frac{x}{a}\sin(ax) + C$$

$$\int x^2\cos(ax)dx = \frac{2x\cos(ax)}{a^2} + \frac{a^2x^2 - 2}{a^3}\sin(ax) + C$$

• Some other integrals of use that are basically functions of x are,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int (a+bx)^{-1} dx = \frac{\ln(a+bx)}{b} + C$$

$$\int (a+bx^2)^{-1} dx = \frac{1}{2\sqrt{(-b)a}} \ln\left(\frac{\sqrt{a} + 2\sqrt{-b}}{\sqrt{a} - x\sqrt{-b}}\right) + C,\ a > 0,\ b < 0$$

$$\int x(a+bx^2)^{-1} dx = \frac{\ln(bx^2+a)}{2b} + C$$

$$\int x^2(a+bx^2)^{-1} dx = \frac{x}{b} - \frac{a}{b\sqrt{ab}} \operatorname{atan}\left(\frac{x\sqrt{ab}}{a}\right) + C$$

$$\int (a^2 - x^2)^{-1} dx = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) + C,\ a^2 > x^2$$

$$\int (a+bx)^{-1} dx = \frac{2\sqrt{a+bx}}{b} + C$$

$$\int x(x^2 \pm a^2)^{-\frac{1}{2}} dx = \sqrt{x^2 \pm a^2} + C$$

$$\int (a+bx+cx^2)^{-1} dx = \frac{1}{\sqrt{c}} \ln\left[\sqrt{a+bx+cx^2} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right] + C,\ c > 0$$

$$\int (a+bx+cx^2)^{-1} dx = \frac{1}{\sqrt{-c}} \operatorname{asin}\left[\frac{-2cx-b}{\sqrt{b^2-4ac}}\right] + C,\ c < 0$$

$$\int (a+bx)^{\frac{1}{2}} dx = \frac{2}{3b}(a+bx)^{\frac{3}{2}}$$

$$\int (a+bx)^{\frac{1}{2}} dx = \frac{2}{3b}(a+bx)^{\frac{3}{2}}$$

$$\int x(a+bx)^{\frac{1}{2}} dx = -\frac{2(2a-3bx)(a+bx)^{\frac{3}{2}}}{15b^2}$$

$$\int (1+a^2x^2)^{\frac{1}{2}} dx = \frac{x(1+a^2x^2)^{\frac{1}{2}} + \frac{\ln\left(x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right)}{a}}{2}$$

$$\int x(1+a^2x^2)^{\frac{1}{2}} dx = \frac{a\left(\frac{1}{a^2}+x^2\right)^{\frac{3}{2}}}{3}$$

$$\int x^2(1+a^2x^2)^{\frac{1}{2}} dx = \frac{ax}{4}\left(\frac{1}{a^2}+x^2\right)^{\frac{3}{2}} - \frac{8}{8a^2}x(1+a^2x^2)^{\frac{1}{2}} - \frac{\ln\left(x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right)}{8a^3}$$

$$\int (1-a^2x^2)^{\frac{1}{2}} dx = \frac{1}{2}\left[x(1-a^2x^2)^{\frac{1}{2}} + \frac{\operatorname{asin}(ax)}{a}\right]$$

$$\int x(1-a^2x^2)^{\frac{1}{2}} dx = -\frac{a}{3}\left(\frac{1}{a^2}-x^2\right)^{\frac{3}{2}}$$

$$\int x^2(a^2-x^2)^{\frac{1}{2}} dx = -\frac{x}{4}(a^2-x^2)^{\frac{3}{2}} + \frac{1}{8}\left[x(a^2-x^2)^{\frac{1}{2}} + a^2\operatorname{asin}\left(\frac{x}{a}\right)\right]$$

$$\int (1+a^2x^2)^{-\frac{1}{2}} dx = \frac{1}{a}\ln\left[x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right]$$

$$\int (1-a^2x^2)^{-\frac{1}{2}} dx = \frac{1}{a}\operatorname{asin}(ax) = -\frac{1}{a}\operatorname{acos}(ax)$$

• Integrals using the natural logarithm base ‘e’,

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1) + C$$

38.3.1 Integration Examples

• Integration by parts - It is normal to have to do the integration by parts more than once to solve a problem.

$$\int x^2 \sin 4x dx$$

$$u =$$

$$v =$$

• Substitution.

$$\int t e^{t^2+1} dt$$

guess,

$$w = t^2 + 1 \qquad \frac{d}{dt}w = 2t \qquad dw = 2t\,dt$$

$$\int t e^{t^2+1} dt = \int e^{t^2+1} t\,dt = \int e^{w} \frac{1}{2} dw = \frac{1}{2} e^{w} + C = \frac{1}{2} e^{t^2+1} + C$$

• Partial fractions can be used to reduce complex polynomials to simple to integrate forms.

$$\int \left(\frac{5x + 10}{x^2 + 3x + 2} \right) dx =$$

38.4 Vectors

• When dealing with large and/or time varying objects or phenomenon we must be able to describe the state at locations, and as a whole. To do this vectors are a very useful tool.

• Consider a basic function and how it may be represented with partial derivatives.

$$y = f(x, y, z)$$

We can write this in differential form, but the right hand side must contain partial derivatives. If we separate the operators from the function, we get a simpler form. We can then look at them as the result of a dot product, and divide it into two vectors.

$$(d)y = \left(\left(\frac{\partial}{\partial x}\right)f(x, y, z)\right)dx + \left(\left(\frac{\partial}{\partial y}\right)f(x, y, z)\right)dy + \left(\left(\frac{\partial}{\partial z}\right)f(x, y, z)\right)dz$$

$$(d)y = \left[\left(\frac{\partial}{\partial x}\right)dx + \left(\frac{\partial}{\partial y}\right)dy + \left(\frac{\partial}{\partial z}\right)dz\right]f(x, y, z)$$

$$(d)y = \left[\left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right) \bullet (dx\underline{i} + dy\underline{j} + dz\underline{k})\right]f(x, y, z)$$

We then replace these vectors with the operators below. In this form we can manipulate the equation easily (whereas the previous form was very awkward).

$$(d)y = [\nabla \bullet dX]f(x, y, z)$$

$$(d)y = \nabla f(x, y, z) \bullet dX$$

$$(d)y = |\nabla f(x, y, z)||dX|\cos\theta$$

In summary,

$$\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

$$F = F_x\underline{i} + F_y\underline{j} + F_z\underline{k}$$

$\nabla \bullet F$ = the divergence of function F

$\nabla \times F$ = the curl of function F

• Gauss's or Green's or divergence theorem is given below. Both sides give the flux across a surface, or out of a volume. This is very useful for dealing with magnetic fields.

$$\int_V (\nabla \bullet F)dV = \oint_A F dA$$

where,

V, A = a volume V enclosed by a surface area A

F = a field or vector value over a volume

• Stoke's theorem is given below. Both sides give the flux across a surface, or out of a volume. This is very useful for dealing with magnetic fields.

$$\int_A (\nabla \times F) dA = \oint_L F dL$$

where,

$A, L =$ A surface area A, with a bounding parimeter of length L

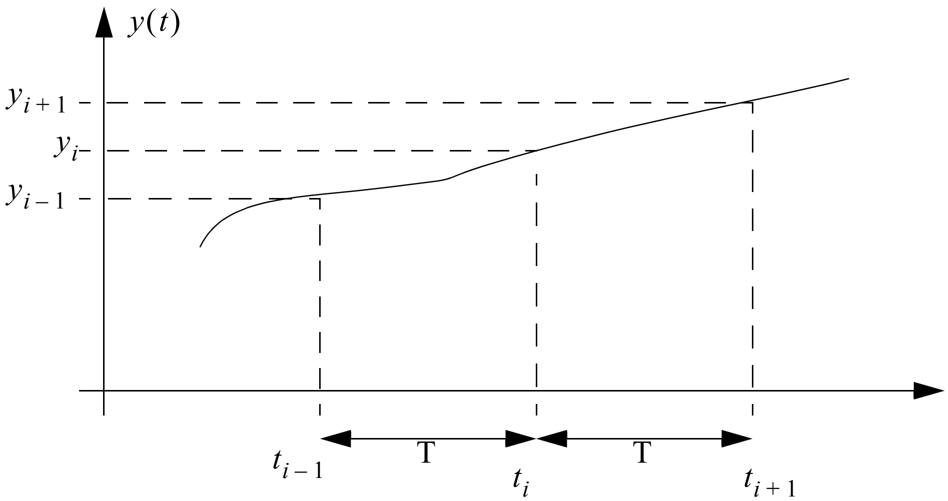
$F =$ a field or vector value over a volume

38.5 Numerical Tools

38.5.1 Approximation of Integrals and Derivatives from Sampled Data

• This form of integration is done numerically - this means by doing repeated calculations to solve the equation. Numerical techniques are not as elegant as solving differential equations, and will result in small errors. But these techniques make it possible to solve complex problems much faster.

• This method uses forward/backward differences to estimate derivatives or integrals from measured data.



$$\int_{t_{i-1}}^{t_i} y(t_i) \approx \left(\frac{y_i + y_{i-1}}{2}\right)(t_i - t_{i-1}) = \frac{T}{2}(y_i + y_{i-1})$$

$$\frac{d}{dt}y(t_i) \approx \left(\frac{y_i - y_{i-1}}{t_i - t_{i-1}}\right) = \left(\frac{y_{i+1} - y_i}{t_{i+1} - t_i}\right) = \frac{1}{T}(y_i - y_{i-1}) = \frac{1}{T}(y_{i+1} - y_i)$$

$$\left(\frac{d}{dt}\right)^2 y(t_i) \approx \frac{\frac{1}{T}(y_{i+1} - y_i) - \frac{1}{T}(y_i - y_{i-1})}{T} = \frac{-2y_i + y_{i-1} + y_{i+1}}{T^2}$$

38.5.2 Centroids and Moments of Inertia

38.6 Problems

1. Find the following derivatives.

a) $\frac{d}{dx}\left(\frac{1}{x+1}\right)$

b) $\frac{d}{dt}(e^{-t}\sin(2t-4))$

ans. $\frac{-1}{(x+1)^2}$

$-e^{-t}\sin(2t-4) + 2e^{-t}\cos(2t-4)$

2. Solve the following integrals.

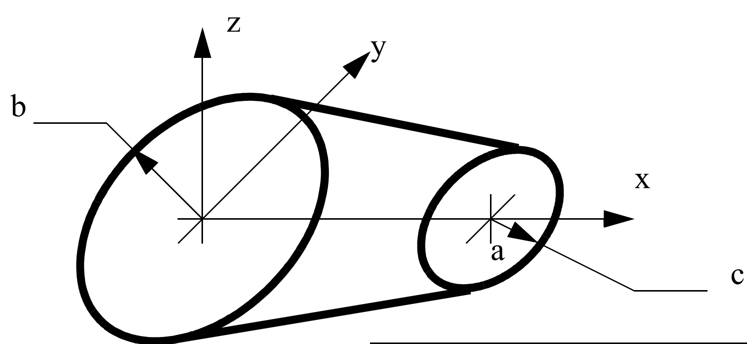
a) $\int e^{2t} dt$

b) $\int (\sin\theta + \cos 3\theta) d\theta$

ans. $\frac{1}{2}e^{2t} + C$

$-\cos\theta + \frac{1}{3}\sin 3\theta + C$

3. Set up an integral and solve it to find the area inside the volume below. The shape is basically a cone with the top cut off.



(ans. $V = \pi\left(\frac{c^2 a}{3} + \frac{b^2 a}{3} + \frac{abc}{3}\right)$

(move later) 4. Write a program that integrates the following function using the trapezoidal rule and Simpson's rule. The period of integration should range from 0 to 10 seconds. The program should compare the numerical results to the exact result.

$$y(t) = 5t^2 + \sin(20t)$$

```
// sample program
function foo = y(t)
        foo = 5 * t ^ 2 + sin ( 20 * t );
endfunction

sum = 0;
h = 0.1; // step size
for t = 0:h:10
        sum = sum + h * (f(t+h) + f(t)) / 2; // uses the trapezoid rule
end

t = 10;
actual = (5 / 3) * t ^ 3 - 0.05 * cos(20 * t);
mprintf("integral value numerical = %f, actual = %f \n", sum, actual);
// the value should be 1667 approximately
```

(move later - near end) 5. Write a program that finds the location of the minimum value for the function given below.

$$y(x) = \sin(\sin(5x^2) + \cos(20x) - 5x) + (x - 10)^2$$

```
// sample program
function foo = y(x)
        foo = sin(sin(5 * x * x) + cos(20 * x) - 5*x) + (x - 10) ^ 2;
endfunction

y_min = 100000000000; // something big to start
x_min = 0; // this value doesn't matter
h = 0.001; // step size - also the accuracy
for x = -100:h:100
        if (y(x) < y_min), // look for the smallest value
        x_min = x;
        y_min = y(x);
        end
end
x_min, y_min
```

8. Differentiate.

a)	$(x^3+4)^4$	(ans. $4(3)x^2(x^3+4)^3$
b)	$(x^3+4)^{-4}$	$-4(3)x^2(x^3+4)^{-5}$
c)	$\ln(x)$	$\frac{1}{x}$
d)	$\ln(x+x^3)$	$\frac{3x^2+1}{x^3+x}$
e)	e^x	e^x
f)	$e^{x+x^2-x^3}$	$(-3x^2+2x+1)e^{x+x^2-x^3}$
g)	$\frac{x}{x^2+5}$	$\frac{-x^2+5}{(x^2+5)^2}$
h)	$\frac{x^2}{x^3+5x}$	$\frac{-x^2+5}{(x^2+5)^2}$
i)	e^{x^2+5}	$2xe^{x^2+5}$
j)	$\sin(x^2)$	$2x\cos x^2$
k)	$e^{x^2}\sin(x)$	$2xe^{x^2}\sin(x)+e^{x^2}\cos(x)$
l)	$\frac{\sin x}{x}$	$\frac{x\cos x-\sin x}{x^2}$

9. Integrate the following indefinite functions with respect to x.

a) $\frac{1}{x}$

b) $\frac{x}{x+5}$

c) $\frac{1}{x^3}$

d) $5x^3 + 2x^2$

e) xe^{x^2}

f) $\cos x$

g) $(\cos x)^2$

(ans.

$\ln(x) + C$

$x - 5\ln(x+5) + C$

$\frac{-1}{2x^2} + C$

$1.25x^4 + \frac{2}{3}x^3 + C$

$\frac{1}{2}e^{x^2} + C$

$\sin x + C$

$\frac{\sin x \cos x + x}{2} + C$

10. Solve the following integrals.

a) $\int_0^\infty e^{-t}dt$

b) $\int_0^{10} (t^2 + 5t^3)dt$

c) $\int_0^{4\pi} \cos(t)dt$

(ans

-1

12833.33

0

11. Write a Scilab program that numerically intergrates the following functions.

a) $\int_0^{10} (t^2 + 5t^3)dt$

b) $\int_0^{4\pi} \cos(t)dt$