

29. ALGEBRA

Topics:

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Objectives:

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29.1 The Fundamentals

29.1.1 Basic Operations

commutative	$a + b = b + a$	
	$a + b = c + d$	$\therefore a = c + d - b$
distributive/collective	$a(b + c) = ab + ac$	
associative	$a(bc) = (ab)c$	$a + (b + c) = (a + b) + c$
	$ab = cd$	$\therefore a = \frac{cd}{b}$
silly mistakes	$\frac{1}{a + b} \neq \frac{1}{a} + \frac{1}{b}$	

- Scilab example,

```

b = 5;
c = 6;
d = 7;
a = c * d / b
a = c + d - b

```

29.1.2 Exponents

- The basic properties of exponents are so important they demand some sort of mention

$(x^n)(x^m) = x^{n+m}$	$x^0 = 1$, if x is not 0	$x^{\frac{1}{n}} = \sqrt[n]{x}$ (nth root)
$\frac{(x^n)}{(x^m)} = x^{n-m} = \frac{1}{x^{m-n}}$	$x^{-p} = \frac{1}{x^p}$	$x^{\frac{m}{n}} = \sqrt[n]{x^m}$
$(x^n)^m = x^{n \cdot m}$	$(xy)^n = (x^n)(y^n)$	$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$
	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	

- Scilab example,

```

x = 5;
n = 2;
m = 4;
y = 6;
x^n * x^m
x^(n + m)
x**(n+m)

```

29.1.3 Basic Polynomials

- The quadratic equation.

$$ax^2 + bx + c = 0 = a(x - r_1)(x - r_2) \quad r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- e.g.,

$$2x^2 + 4x + 2 = 0$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(2)}}{2(2)} = -1 \pm \sqrt{0} = -1, -1$$

$$\therefore (x+1)(x+1) = 0 \quad \leftarrow \text{note the signs}$$

- Complex roots will occur when,

$$b^2 - 4ac < 0$$

- Cubic equations can be solved explicitly, although this is too much for common memorization.

$$x^3 + ax^2 + bx + c = 0 = (x - r_1)(x - r_2)(x - r_3)$$

First, calculate,

$$Q = \frac{3b - a^2}{9} \quad R = \frac{9ab - 27c - 2a^3}{54} \quad S = \sqrt[3]{R + \sqrt{Q^3 + R^2}} \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

Then the roots,

$$r_1 = S + T - \frac{a}{3} \quad r_2 = \frac{S+T}{2} - \frac{a}{3} + \frac{j\sqrt{3}}{2}(S-T) \quad r_3 = \frac{S+T}{2} - \frac{a}{3} - \frac{j\sqrt{3}}{2}(S-T)$$

- On a few occasions a quartic (4th order) equation will also have to be solved. This can be done by first reducing the equation to a quadratic,

$$x^4 + ax^3 + bx^2 + cx + d = 0 = (x - r_1)(x - r_2)(x - r_3)(x - r_4)$$

First, solve the equation below to get a real root (call it 'y'),

$$y^3 - by^2 + (ac - 4d)y + (4bd - c^2 - a^2d) = 0$$

Next, find the roots of the 2 equations below,

$$r_1, r_2 = z^2 + \left(\frac{a + \sqrt{a^2 - 4b + 4y}}{2} \right) z + \left(\frac{y + \sqrt{y^2 - 4d}}{2} \right) = 0$$

$$r_3, r_4 = z^2 + \left(\frac{a - \sqrt{a^2 - 4b + 4y}}{2} \right) z + \left(\frac{y - \sqrt{y^2 - 4d}}{2} \right) = 0$$

• In Scilab,

```
x = poly(0, 'x');
roots(3 * x^2 + 4 * x + 2)
```

```
q = [2, 4, 3];
p = poly(q, 'x', 'coeff'); // defines a polynomial using a vector
roots(p)
```

```
derivative(p, 'x'); // finds the derivative of the polynomial
horner(p, 5); // evalautes the polynomial at 5
s = (x + 1) / p; // an algebra operation
```

29.2 Special Forms

29.2.1 Completing the Square

$$\begin{aligned}x^2 + Ax + B &= (x + C)^2 + D \\ &= x^2 + 2Cx + (C^2 + D)\end{aligned}$$

$$A = 2C \qquad B = C^2 + D$$

$$C = \frac{A}{2} \qquad D = B - C^2$$

for example, given,

$$\begin{aligned}5x^2 + 50x + 10 \\ &= 5(x^2 + 10x + 2) \\ &= 5((x + 5)^2 - 23)\end{aligned}$$

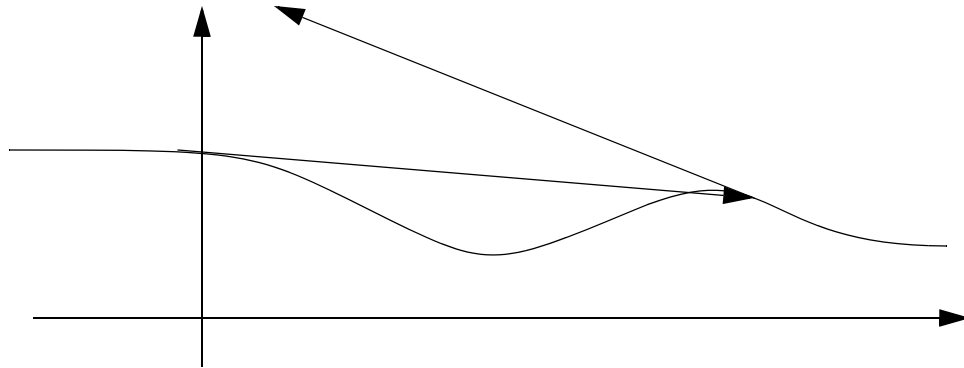
29.2.2 Newton-Raphson to Find Roots

- When given an equation where an algebraic solution is not feasible, a numerical solution may be required. One simple technique uses an instantaneous slope of the function, and takes iterative steps towards a solution.

$$x_{i+1} = x_i - \frac{f(x_i)}{\left(\frac{d}{dx}f(x_i)\right)}$$

- The function $f(x)$ is supplied by the user along with an initial guess.
- This method can become divergent if the function has an inflection point near the root.

- The technique is also sensitive to the initial guess.



- This calculation should be repeated until the final solution is found.
- Scilab example,

NEWTON RAPHSON ROOT EXAMPLE

29.3 Complex Numbers

- Complex values

The Complex (imaginary) Number:

$$j = \sqrt{-1} \qquad j^2 = -1$$

Complex Numbers:

$$a + bj \qquad \text{where, } a \text{ and } b \text{ are both real numbers}$$

Complex Conjugates (denoted by adding an asterisk '*' to the variable):

$$N = a + bj \qquad N^* = a - bj$$

Basic Properties:

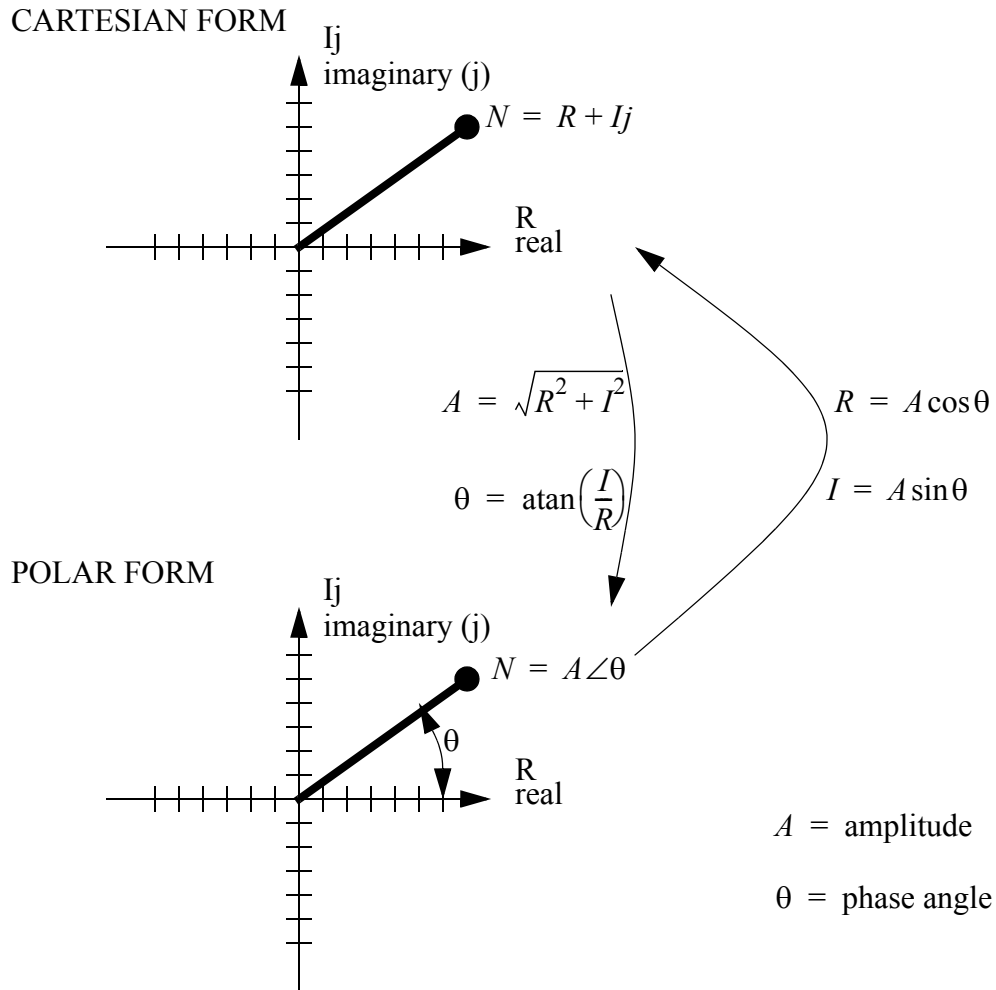
$$(a + bj) + (c + dj) = (a + c) + (b + d)j$$

$$(a + bj) - (c + dj) = (a - c) + (b - d)j$$

$$(a + bj) \cdot (c + dj) = (ac - bd) + (ad + bc)j$$

$$\frac{N}{M} = \frac{a + bj}{c + dj} = \frac{N(N^*)}{M(N^*)} = \frac{(a + bj)(c - dj)}{(c + dj)(c - dj)} = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)j$$

- We can also show complex numbers graphically. These representations lead to alternative representations. If it is not obvious above, please consider the notation uses a cartesian notation, but a polar notation can also be very useful when doing large calculations.



- We can also do calculations using polar notation (this is well suited to multiplication and division, whereas cartesian notation is easier for addition and subtraction),

$$A \angle \theta = A(\cos \theta + j \sin \theta) = A e^{j\theta}$$

$$e^{A+jB} = e^A e^{jB} = e^A (\cos \theta + j \sin \theta)$$

$$(A_1 \angle \theta_1)(A_2 \angle \theta_2) = (A_1 A_2) \angle (\theta_1 + \theta_2)$$

$$\frac{(A_1 \angle \theta_1)}{(A_2 \angle \theta_2)} = \left(\frac{A_1}{A_2}\right) \angle (\theta_1 - \theta_2)$$

$$(A \angle \theta)^n = (A^n) \angle (n\theta) \quad (\text{DeMoivre's theorem})$$

- Note that DeMoivre's theorem can be used to find exponents (including roots) of complex numbers
- Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

Note: for $0 + 1j = \cos\theta + j\sin\theta$

$$\theta = \frac{\pi}{2}$$

$$e^{j\frac{\pi}{2}} = j$$

- From the above, the following useful identities arise:

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- In Scilab

```

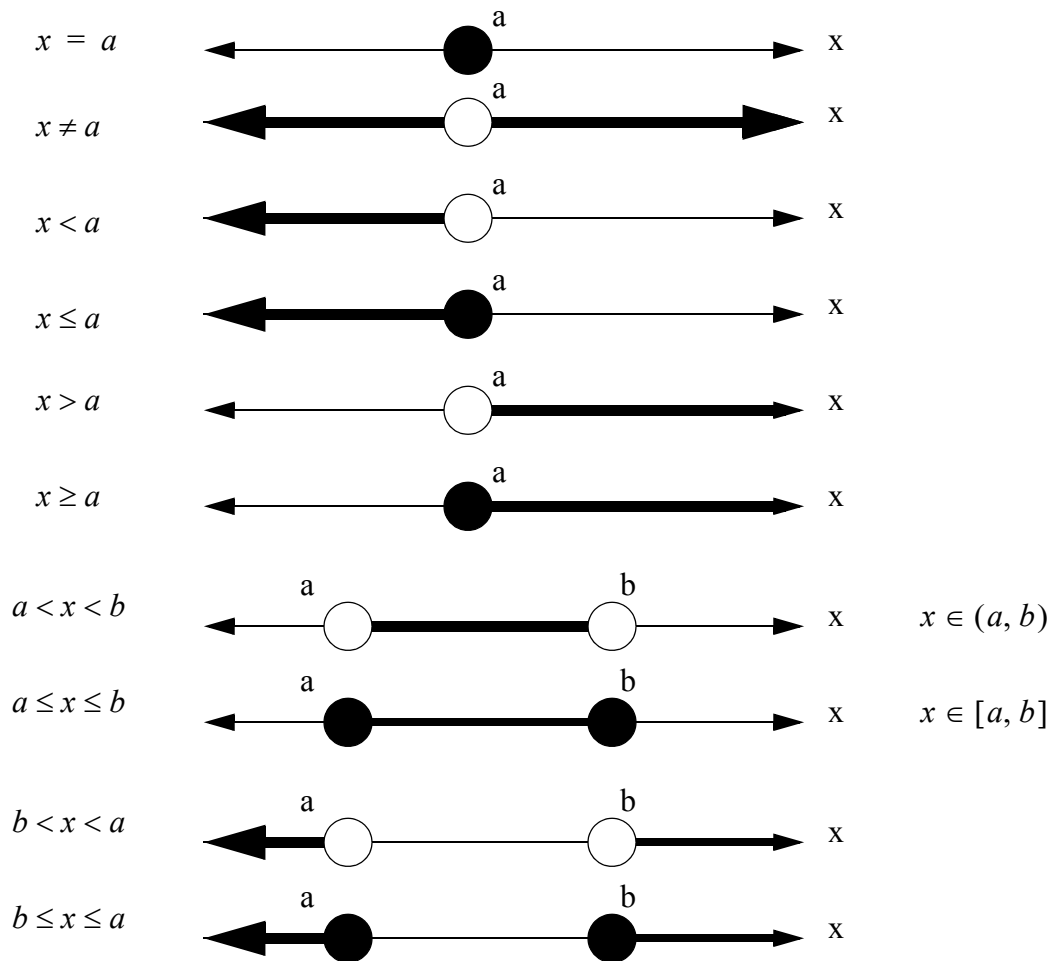
j = sqrt(-1);
A = (1 + 2 * j) / (3 + 4 * j);
A
[mag, theta] = polar(A);
mag, theta // the magnitude and angle of the complex angle A
abs(A); // the magnitude of the complex number
real(A); // the real part of the complex number
imag(A); // the imaginary part of the complex number
conj(A); // the conjugate of the complex number
atan(imag(A), real(A)); // the angle of the complex number

```

29.4 Equality and Inequality

- Some basic relationships are illustrated below with number lines. The shaded dots indicate that

the values include the point. Unshaded dots indicate that the values approach but do not equal the value.



- rearranging equations with inequalities

$$a + b < 5 \quad \therefore a < 5 - b$$

- When doing calculations there is some roundoff error that result in a number that should be zero but has a finite value. Consider single precision floating point numbers with 7 digits, or double precision with 14, the last digit is equality in numerical calculations using a tolerance. In these

cases a tolerance is used between the target and actual values to determine equality.

$$\varepsilon = |val| - \text{target}$$

$$\text{if}(\varepsilon < 0.000001) \text{ then } \varepsilon = 0$$

- In Scilab,

```
if ( x > A ) | ( x < B ) then
    // program statement
else if x == A then
    // more program statements
end
```

- Another Scilab example,

$$F(x) = \begin{cases} 5 & 0 < x \leq 5 \\ 0 & x \leq 0, x > 5 \end{cases}$$

```
function foo = F(x)
    if (x > 0) & (x <= 5) then
        foo = 5;
    else
        foo = 0;
    end
end_function
```

29.5 Functions

- Functions help to identify and organize self contained expressions.
- defining functions encourages reuse of mathematical expressions or program code
- terminology,

$$f(x) \quad (\text{say 'f of x'})$$

- an example function is,

$$f(x) = x^2 + 3$$

$$f(2) = 2^2 + 3 = 7$$

- In Scilab the function becomes,

```
function foo = f(x)
    foo = x^2 + 3;
endfunction

mprintf("%f\n", f(2));
```

Note: Use indents when writing programs to ensure structure. While these are not necessary they will make debugging much easier.

Note: Using functions is not necessary in simpler programs, but larger programs will become very difficult to work with if functions are not used.

29.6 Special Functions

29.6.1 Logarithms

- Logarithms also have a few basic properties of use,

The basic base 10 logarithm:

$$\log x = y \qquad x = 10^y$$

The basic base n logarithm:

$$\log_n x = y \qquad x = n^y$$

The basic natural logarithm (e is a constant with a value found near the start of this section):

$$\ln x = \log_e x = y \qquad x = e^y$$

- All logarithms observe a basic set of rules for their application,

$$\log_n(xy) = \log_n(x) + \log_n(y)$$

$$\log_n(n) = 1$$

$$\log_n\left(\frac{x}{y}\right) = \log_n(x) - \log_n(y)$$

$$\log_n(1) = 0$$

$$\log_n(x^y) = y\log_n(x)$$

$$\log_n(x) = \frac{\log_m(x)}{\log_m(n)}$$

$$\ln(A \angle \theta) = \ln(A) + (\theta + 2\pi k)j \quad k \in I$$

e.g.,

solve

$$5 = 10^n$$

$$\log(5) = n\log(10)$$

$$n = \frac{\log(5)}{\log(10)}$$

- Note: most computers use the 'log' function for the natural logarithm. The base 10 logarithm is normally 'log10'.

- In Scilab this is

log10(3) - base 10 log

log(3) - natural log

log(exp(10))

29.7 Solving Systems of Linear Equations

- Systems of linear equations are of the form below,

$$x + y = 5$$

$$2x + 3y = 8$$

- In general there must be the same number of equations and unknowns to solve the equations.
- In some cases the equations will not be solvable. In this case we say the equations are singular.

29.7.1 Substitution

- Substitution is the most fundamental method for solving linear equations, but it is the least routine,

Given,

$$x + y = 5$$

$$2x + 3y = 8$$

To solve for y, we substitute to eliminate x.

$$\therefore x = 5 - y$$

$$\therefore 2(5 - y) + 3y = 8$$

$$\therefore y = -2$$

To solve for x, we substitute the value for y into an earlier equation.

$$\therefore x = 5 - (-2) = 7$$

- In Scilab this is

EXAMPLE ????????????

29.7.2 Addition

- Polynomials can be added to eliminate variables (in a matrix for this is referred to as the Gauss-Jordan row reduction method).

Given,

$$x + y = 5$$

$$2x + 3y = 8$$

multiply and add the equations,

$$(-2)(x + y = 5)$$

$$2x + 3y = 8$$

$$\hline 0x + 1y = -2$$

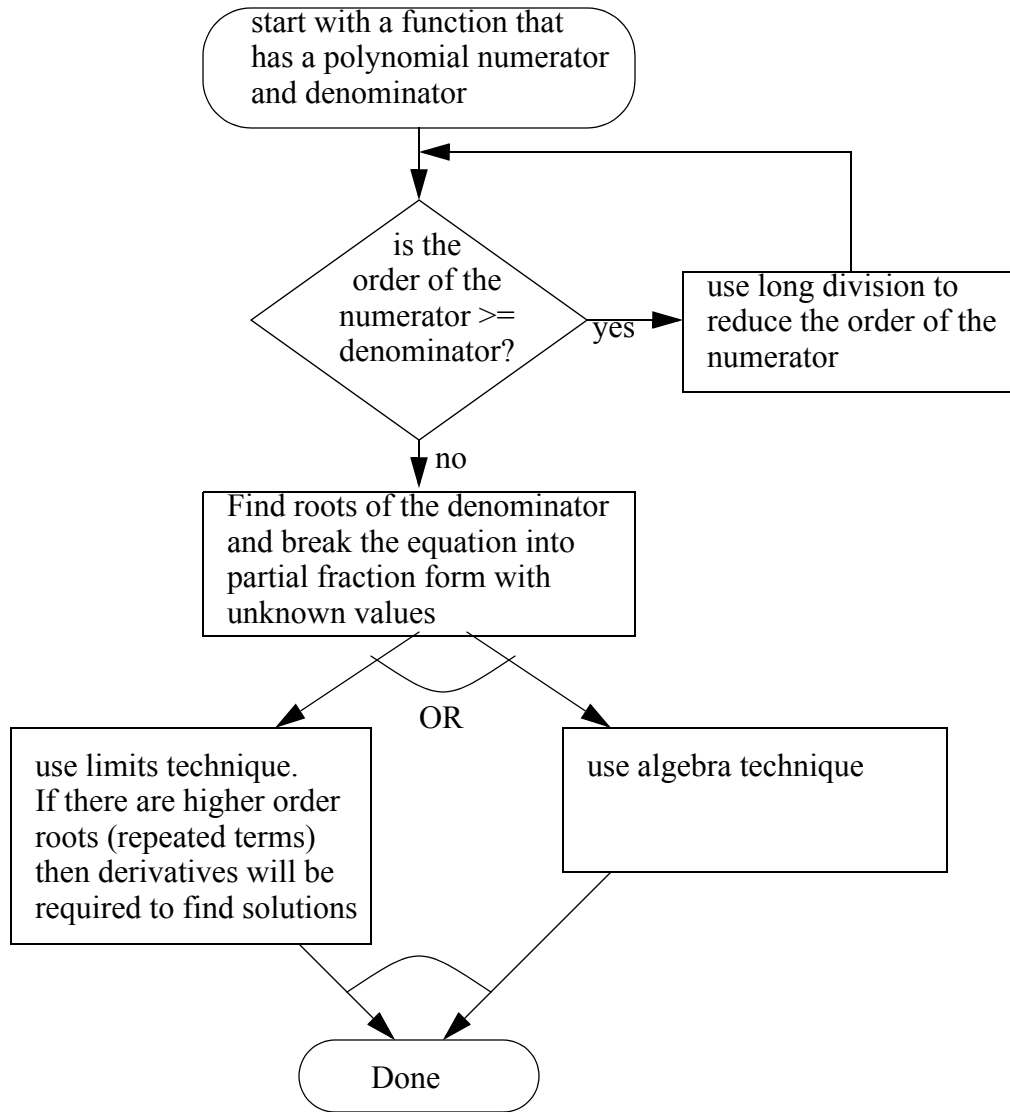
$$\therefore y = -2$$

$$x = 5 - (-2) = 7$$

29.8 Simplifying Polynomial Expressions

29.8.1 Partial Fractions

- The next is a flowchart for partial fraction expansions.



- The partial fraction expansion for,

$$x(s) = \frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$C = \lim_{s \rightarrow -1} \left[(s+1) \left(\frac{1}{s^2(s+1)} \right) \right] = 1$$

$$A = \lim_{s \rightarrow 0} \left[s^2 \left(\frac{1}{s^2(s+1)} \right) \right] = \lim_{s \rightarrow 0} \left[\frac{1}{s+1} \right] = 1$$

$$B = \lim_{s \rightarrow 0} \left[\frac{d}{ds} \left[s^2 \left(\frac{1}{s^2(s+1)} \right) \right] \right] = \lim_{s \rightarrow 0} \left[\frac{d}{ds} \left(\frac{1}{s+1} \right) \right] = \lim_{s \rightarrow 0} [-(s+1)^{-2}] = -1$$

- Consider the example below where the order of the numerator is larger than the denominator.

$$x(s) = \frac{5s^3 + 3s^2 + 8s + 6}{s^2 + 4}$$

This cannot be solved using partial fractions because the numerator is 3rd order and the denominator is only 2nd order. Therefore long division can be used to reduce the order of the equation.

$$\begin{array}{r}
 5s + 3 \\
 \hline
 s^2 + 4 \overline{) 5s^3 + 3s^2 + 8s + 6} \\
 \underline{5s^3 + 20s} \\
 3s^2 - 12s + 6 \\
 \underline{3s^2 + 12} \\
 -12s - 6
 \end{array}$$

This can now be used to write a new function that has a reduced portion that can be solved with partial fractions.

$$x(s) = 5s + 3 + \frac{-12s - 6}{s^2 + 4} \quad \text{solve} \quad \frac{-12s - 6}{s^2 + 4} = \frac{A}{s + 2j} + \frac{B}{s - 2j}$$

- When the order of the denominator terms is greater than 1 it requires an expanded partial fraction form, as shown below.

$$F(s) = \frac{5}{s^2(s+1)^3}$$

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

- We can solve the previous problem using the algebra technique.

$$\begin{aligned} \frac{5}{s^2(s+1)^3} &= \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)} \\ &= \frac{A(s+1)^3 + Bs(s+1)^3 + Cs^2 + Ds^2(s+1) + Es^2(s+1)^2}{s^2(s+1)^3} \\ &= \frac{s^4(B+E) + s^3(A+3B+D+2E) + s^2(3A+3B+C+D+E) + s(3A+B) + (A)}{s^2(s+1)^3} \end{aligned}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \quad \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \\ 5 \\ 10 \\ 15 \end{bmatrix}$$

$$\frac{5}{s^2(s+1)^3} = \frac{5}{s^2} + \frac{-15}{s} + \frac{5}{(s+1)^3} + \frac{10}{(s+1)^2} + \frac{15}{(s+1)}$$

29.8.2 Summation and Series

- The notation $\sum_{i=a}^b x_i$ is equivalent to $x_a + x_{a+1} + x_{a+2} + \dots + x_b$ assuming a and b are integers and $b \geq a$. The index variable i is an index often replaced with j , k , m , and n .

- Operations on summations:

$$\sum_{i=a}^b x_i = \sum_{i=b}^a x_i$$

$$\sum_{i=a}^b \alpha x_i = \alpha \sum_{i=a}^b x_i$$

$$\sum_{i=a}^b x_i + \sum_{j=a}^c y_j = \sum_{i=a}^c (x_i + y_i)$$

$$\sum_{i=a}^b x_i + \sum_{i=b+1}^c x_i = \sum_{i=a}^c x_i$$

$$\left(\sum_{i=a}^b x_i \right) \left(\sum_{j=c}^d y_j \right) = \sum_{i=a}^b \sum_{j=c}^d x_i y_j$$

- Some common summations:

$$\sum_{i=1}^N i = \frac{1}{2}N(N+1)$$

$$\sum_{i=0}^{N-1} \alpha^i = \begin{cases} \frac{1-\alpha^N}{1-\alpha}, & \alpha \neq 1 \\ N, & \alpha = 1 \end{cases} \text{ for both real and complex } \alpha.$$

$$\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}, \quad |\alpha| < 1 \text{ for both real and complex } \alpha. \text{ For } |\alpha| \geq 1, \text{ the summation does not converge.}$$

- In Scilab

LOOP EXAMPLE

29.9 Limits

- Limits stuff goes here.....

29.10 Problems

1. Rearrange the following equation so that only 'y' is on the left hand side.

$$\frac{y+x}{y+z} = x+2 \qquad \text{ans.} \qquad y = \frac{-x+xz+2z}{-x-1}$$

2. Solve the following equation to find 'x'.

$$2x^2 + 8x = -8 \qquad \text{ans. } x = -2, -2$$

3. Solve the following system of equations using substitution.

$$x + 2y + 3z = 5$$

$$\text{ans. } x=-7, y=18.75, z=-8.5$$

$$x + 4y + 8z = 0$$

$$4x + 2y + z = 1$$

4. Simplify the following expressions.

a) $n\log(x) + m\log(y) - \log(z)$

$$\text{ans. } \log\left(\frac{x^n y^m}{z}\right)$$

5. Simplify,

$$(5x + 3y) - (2x - 7y)$$

$$(5x^2 + 3y) - (2x - 7y)$$

$$\frac{x^2 + 5x + 6}{x^2 + 7x + 12}$$

$$\frac{5}{x} + \frac{7}{3x}$$

$$\frac{5}{x} - \frac{7}{3x}$$

$$\frac{5}{3 - \frac{7}{x}}$$

$$\frac{5}{3 - \frac{7}{x}}$$

$$5(x + 3) - 9(x - 6)$$

$$\frac{5}{x-2} = \frac{6}{x+3}$$

$$\text{ans. } 3x + 10y$$

$$5x^2 - 2x + 4y$$

$$\frac{x+2}{x+4}$$

$$\frac{22}{3x}$$

$$\frac{8}{3x}$$

$$\frac{5x}{3x-7}$$

$$-4x + 69$$

$$x = 27$$

6. Multiply,

$$(2x + 3)(5x + 4)$$

$$(2x^2 + 3x + 4)(5x + 4)$$

$$\text{ans. } 10x^2 + 23x + 12$$

$$10x^3 + 23x^2 + 32x + 16$$

7. Factor,

$$x^2 - 4$$

$$x^2 - 4y^2$$

$$x^2 - 4x + 4$$

$$x^2 + 5x + 6$$

ans.	$(x + 2)(x - 2)$
	$(x + 2y)(x - 2y)$
	$(x - 2)(x - 2)$
	$(x + 2)(x + 3)$

10. solve for x, y,

$$5x + 3y = 10$$

$$x - y = 0$$

ans. $x=y=1.25$

11. Find roots,

$$x^2 + 2x + 100 = 0$$

$$x^4 + 2x^2 + 100 = 0$$

ans.	$-1 \pm \frac{\sqrt{-396}}{4}$
	$x = \pm \sqrt{-1 \pm \frac{\sqrt{-396}}{4}}$

13. Convert the following log to base 10,

$$\log_6 3$$

ans. 0.613

14. Find x,

$$x^{12} = 1372$$

ans. $x = 1.826$

$$5^{12+x} = 1372$$

15. Simplify,

$$\frac{\log A + \log B}{C \log D}$$

ans. $\frac{1}{C} \log(AB - D)$

16. Find the polar form for,

$$5 + 6j$$

ans. $7.81 \angle 0.876 \text{rad}$

17. Find the cartesian complex value for,

$$10 \angle 0.13 \text{rad}$$

ans. $9.92 + 1.30j$

18. Find the cartesian complex value for,

$$(10 \angle 0.13 \text{rad})^3$$

ans. $924.9 + 380.2j$

19. Solve,

$$-5x - 2 > 3x + 4$$

$$x^2 - 36 > 0$$

$$x^2 + 36 > -12x$$

ans. $x < -0.75$

$$|x| > 6$$

$$x >$$

20. Solve,

$$(8 - 3j) - (12 + 4j)$$

$$(4 + 5j)^2$$

$$\left(\frac{8 - 3j}{4 + 5j}\right)$$

$$(x - 6 - 4j)(x - 6 + 4j)$$

ans. $-4 - 7j$

$$-9 - 40j$$

$$0.415 - 1.268j$$

$$x^2 - 12x + 52$$

21. Find the limits below.

a) $\lim_{t \rightarrow 0} \left(\frac{t^3 + 5}{5t^3 + 1} \right)$ ans. 5

b) $\lim_{t \rightarrow \infty} \left(\frac{t^3 + 5}{5t^3 + 1} \right)$ ans. 1/5

22. Evaluate the limits,

$$\lim_{t \rightarrow \infty} \frac{\sin(t)}{t}$$

$$\lim_{t \rightarrow 0} \frac{\sin(t)}{t}$$

$$\lim_{t \rightarrow \infty} \sin(t)$$

$$\lim_{t \rightarrow 0} \sin(t)$$

$$\lim_{t \rightarrow \infty} te^{-t}$$

$$\lim_{t \rightarrow 0} te^{-t}$$

ans.

23. Reduce the following expression to partial fraction form.

$$\frac{x^2 + 4}{x^3 + 6x^2 + 9x}$$

ans. $\frac{4}{9x} - \frac{13}{3(x+3)^2} + \frac{5}{9(x+3)}$

29.11 Challenge Problems

1. Write a program (in Scilab, C, etc.) that will perform the following calculation.

$$y(x) = \sum_{i=1}^n (x+i)^2$$

2. Develop a program to fit a polynomial to set set of points. In the case where the order of the polynomial does not allow an exact fit, a least squares method should be used to obtain the best fit.

3. Write a program in Scilab (ask if you would prefer to use another platform). The program should use the Newton-Raphson method to find the zeros of an arbitrary function. The functions below should be used for testing the program. The final program will be tested for robustness. The final program should be structured.

$$x^2 + x + 5 = 0$$

$$-5x^3 + x^2 + 10 = 0$$